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Integral Representations over Isotropic Submanifolds and Equations of Zero Curvature

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In the phase space over a Riemann manifold we consider a submanifold Λ invariant with respect to a Hamilton flow, isotropic (i.e., the form $pdq|_\Lambda$ is closed),

Gaussian packets framed by an "amplitude." The amplitude is a parallel section of a bundle of polynomials. The connection on this bundle is generated by a symplectic connection with zero curvature on the normal symplectic bundle over Λ. The coefficients of such a connection are calculated explicitly in terms of infinitesimal symmetries, and also in adiabatic approximation. We investigate topological and geometrical objects arising as corrections to the Poincaré–Cartan invariant in quantization rule on Λ and calculate spectral series for the quantum Hamiltonian. © 1998 Academic Press

1. INTRODUCTION

In this paper we consider the equation

$$H(q, -i\hbar\, \partial/\partial q)\, \psi(q) = \lambda \psi(q), \qquad \hbar \to 0 \tag{1.1}$$

over a Riemann manifold $\mathscr{M}$. We assume that the Hamilton system on $T^*\mathscr{M}$, corresponding to the function $H(q, p)$, has an invariant isotropic submanifold

$$\Lambda = \{q = q(\alpha),\, p = p(\alpha)\} \subset \{H(q, p) = \text{const}\},$$

and has an invariant measure $d\sigma$ on Λ. We also suppose that the linearized Hamilton flow is stable over Λ (in the sense, explained below). In this situation we shall construct a global asymptotic solution (*quasimodes*) for problem (1.1) by using the ansatz

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$$\psi(q) = \frac{1}{c(\hbar)} \int_{\Lambda} \tilde{g}\left(\alpha, \frac{q - q(\alpha)}{\sqrt{\hbar}}\right)$$
$$\times \exp\left\{\frac{i}{\hbar} p(\alpha) \cdot (q - q(\alpha)) + \frac{1}{\hbar} \int_{\alpha_0}^{\alpha} p(\alpha') \cdot dq(\alpha')\right\} d\sigma(\alpha). \quad (1.2)$$

Here $\alpha \in \Lambda$, and $c(\hbar)$ is a normalization constant, the function $\tilde{g}$ is defined on a vector bundle over Λ. This function will have the form

$$\tilde{g}(\alpha, y) = \exp\left\{\frac{i}{2} A(\alpha)\, y \cdot y\right\} \tilde{\varphi}(\alpha, y), \quad (1.2a)$$

where $A(\alpha)$ are symmetric matrices, $\operatorname{Im} A(\alpha) > 0$, and $\tilde{\varphi}(\alpha, \cdot)$ are certain *polynomials* over the fibers of the vector bundle.

Our aim is to investigate the ansatz (1.2) and to introduce certain geometrical and analytical objects useful not only for the problem (1.1), but for other topics in symplectic and quantum geometry.

The developing of a technique for construction of quasimodes (may be without explicit using the small parameter $\hbar$) is one of the fundamental directions in the theory of semiclassical approximation and in the general theory of pseudodifferential operators [2, 4, 26, 33, 34, 69, 82]. And it is also the base for different theories of quantization [46, 46a, 54, 56, 80, 91].

First steps in construction of quasimodes in the multidimensional case were taken by J. B. Keller [53]. The quasimodes, whose front of oscillations coincides with a given *Lagrangian* submanifold, were first introduced by V. P. Maslov [66] by using local WKB-asymptotics and their Fourier transformations in charts on Λ and matching them together on the intersections by the stationary phase method, what lead to a quantization rule of the Bohr-Sommerfeld type with the general Maslov cohomology class. Similar constructions from the general point of Lagrangian Grassmannian were proposed by V. S. Buslaev [15] and J. Leray [60]. For the case of homogeneous Lagrangian submanifolds the fundamental investigation of quasimodes ("Fourier integral operators") appeared in the L. Hörmander and J. J. Duistermaat papers [22, 37]. Deep results in this field were obtained later by many mathematicians; see review in [29]. Specially for integrable systems (where a total family of invariant Lagrangian submanifolds is generated by action-angle variables) and for their perturbations a remarkable technique of construction of quasimodes was developed by A. Weinstein, V. Guillemin, S. Sternberg, L. Boutet de Monvel, Y. Colin de Verdiere [14, 17, 30, 88].

And what happens if we have not a family, but a single invariant submanifold, which is not Lagrangian, but only isotropic? The notion "*isotropic*" means that the form $p\,dq$ is closed on Λ, but perhaps, $\dim \Lambda > \dim \mathscr{M}$. Such submanifolds of lower dimension invariant with respect to the Hamilton flow play an important role in applications and appear even more often than Lagrangian submanifolds, since they are "closer" to the general nonintegrable case. As an example we can mention isolated geodesics, invariant tori (or topologically more complicated submanifolds) for Hamilton systems with infinitesimal symmetries, systems with Dirac constraints, etc. [5, 11, 29, 59, 64, 70, 72, 79].

In the isotropic case the difference of dimensions $\dim \mathscr{M} - \dim \Lambda$ yields degrees of freedom allowing the trajectories of the Hamilton system to go out of a neighborhood of Λ after an infinitesimal deviation of the initial point from Λ. If this phenomenon does not take place, i.e., in other words, if the first variation equation of the Hamilton field is stable on Λ, then local WKB-asymptotics can be constructed by means of Gaussian packets or excited oscillator modes decreasing along the directions normal to Λ. This ansatz was proposed by V. M. Babich with collaborators [6, 7] for the case of closed geodesics when $\dim \Lambda = 1$ (see also discussions in [32, 85]). In the context of general symplectic geometry and Gelfand–Lidsky index, the case $\dim \Lambda = 1$ was investigated very carefully [16, 31, 75]. When $\dim \Lambda \geqslant 1$ and Λ is a *homogeneous* isotropic submanifold, a very interesting ansatz for quasimodes was proposed in [28]. For a general nonhomogeneous case quasimodes were constructed in [67] by means of an additional complex Lagrangian subbundle over Λ ("complex germ"), see also [8, 20, 58]. In the work [8], the quasimodes were calculated using an invariant trivialization of the complex germ over the universal covering of Λ. But all these constructions were based on cumbersome matchings of local WKB-asymptotics on intersections of charts.

In the given paper we do not use any matching. We work with a global integral representation (1.2) for quasimodes proposed in the paper [40] (see also [41–44], and further generalizations [44a, 45]). The ordinary difficulties and obstructions (singularities of the projection of Λ on $\mathscr{M}$, e.g., focal points, caustics, etc.) does not appear in this representation. Only a type of symplectic embedding of Λ into $T^*\mathscr{M}$ is important. The new ansatz gives a possibility to construct the quasimodes in general situation which cannot be served by the technique mentioned above, and it leads to consideration of very interesting geometrical objects over an isotropic submanifold.

This representation is close to the analytical Bargmann–Klauder transformation [55, 86], but employs the *integral along a classical invariant submanifold* Λ, not the integral over a complex phase space. Physicists feel that such representation could be very convenient for quantization. Apparently, E. J. Heller [35] was the first who used it (for one-dimensional oscillator, and then in heuristic way for chaotic systems by using the integration of Gaussian packets along the classical trajectory); in this direction see also [62, 68, 74, 73a].

The principal idea in [40–44] was to use a linear connection over a classical invariant submanifold Λ in order to generate convenient quantum packets by parallel transport. “Convenient” means that the packet is a kernel of intertwining map between the initial representation of quantum algebra on the space $L^2(\mathscr{M})$ and a new representation on the space of functions over Λ (or on the space of sections of a certain linear bundle over Λ as below). So, Λ plays here the role of the “configuration” space for a new quantum representation. The operators of the new representation are very simple in the semiclassical approximation:

$$H|_{\Lambda} - i\hbar D_H + \frac{\hbar}{2}\langle \varkappa, \operatorname{ad}(H)\rangle.$$

Here $\operatorname{ad}(H)$ is the Hamilton field corresponding to $H(q, p)$, D_H is a vector field on a linear bundle of polynomials over Λ, and $\varkappa$ is a “gauge” 1-form on Λ (see Theorem 4.1 below). Operators of such type are used, for instance, in the Segal quantization [76]. So, from the initial quantization $H \to H(q, -i\hbar\, \partial/\partial q)$ over $\mathscr{M}$, we come mod $O(\hbar^2)$ to the quantization over an invariant stable isotropic submanifold $\Lambda \subset T^*\mathscr{M}$.

Note that this construction correlates with the well-known Blattner–Kostant–Sternberg pairing in the theory of geometric quantization [9, 54, 56] and with the technique of symplectic connections [36, 73]. But it is important to stress that we consider a single invariant submanifold Λ, not a family. In general position, when $1 < \dim \Lambda < \dim \mathscr{M}$, there is no family of invariant isotropic submanifolds near the given Λ, i.e., an analog of the Poincaré theorem about a family of invariant curves is absent. That is the reason that we must work only with infinitesimal objects over Λ, and cannot use the geometric quantization technique (for example, use integrable Kähler polarizations, pairing of polarizations and standard Maslov class, etc.)

There is also close correlation with Guillemin’s symplectic spinors [28]. In fact, the function (1.2a) is an example of such a spinor. So, we use special subspace in the total spinor space over isotropic Λ. In this sense our ansatz (1.2) can be considered as a detailing and at the same time a “globalization” of the local one studied in [28].

In the case of a Lagrangian submanifold Λ (i.e., $\dim \Lambda = \dim \mathscr{M}$), a global formula of type (1.2), proposed in [40], uses the amplitude $\tilde{\varphi}$ (1.2a), which does not depend on y and is given by the expression

$$\tilde{\varphi}(\alpha) = |J(\alpha)|^{1/2} \exp\left\{-\frac{i\pi}{2}\int_{\alpha_0}^{\alpha} \mu\right\}.$$

Here

$$J(\alpha) \stackrel{\text{def}}{=} \det\left(\frac{\partial p(\alpha)}{\partial \alpha} - A(\alpha)\frac{\partial q(\alpha)}{\partial \alpha}\right) \tag{1.3}$$

and the form μ is defined on Λ by the formula

$$\mu = -\left(\frac{J^2}{|J|^2}\right)^* \phi, \tag{1.4}$$

ϕ is a fundamental form on $\mathbf{S}^1$, $\oint_{\mathbf{S}^1} \phi = 1$.

It was long known [3, 19, 29, 81] that the cohomology class of the Arnold form, like (1.4), coincides with the Maslov class of the Lagrangian submanifold. Now it turned out that the form μ itself takes part in the expression for the *global phase of a quasimode*.

In the given paper for the case of an isotropic submanifold Λ we shall use the same ansatz (1.2). But the Jacobian J cannot be already defined by formula (1.3). We show how to calculate the modulus and the phase of the amplitude $\tilde{\varphi}$ by a certain additional Hermitian subbundle τ over Λ. The phase is given by the *gauge form* associated with the bundle τ. We define this form and show that it consists of topological and dynamical summands. The topological one leads to an integral cohomology class over Λ, and the dynamical one coincides with the connection form on τ. These results were announced in [47–50]; an initial version of the given paper was published as preprint [52].

For the amplitude $\tilde{\varphi}$ under the integral (1.2), (1.2a), we obtain an equation of parallel transport in the bundle $\mathrm{Pol}(\bar{\tau})$ of polynomial functions on the fibers of $\bar{\tau}$. The solutions of this equation can be the generalized *Hermite polynomials*, as well as, some *other* interesting polynomials. The latter appear if the group $\pi_1(\Lambda)$ is noncommutative; the coefficients of these polynomials depend on the holonomy of the Hermitian connection on τ. We also do not exclude the case of a nonorientable submanifold Λ, which is important for applications. As an example, we consider polynomials on the bundle over the Klein bottle $\mathbf{K}^2$.

The curvature of the Hermitian connection on τ can be either equal to zero, or not. Both variants are interesting. The case of zero curvature is the simplest from the viewpoint of structure (1.2) of a wave function. At the same time the existence of a connection with zero curvature which is *adapted to the first variation equation* of a given Hamiltonian field, is an additional difficult problem. (An analog of this problem in the theory of integrable systems is the existence of the Lax pair).

We stress that this problem does not appear in the case $\dim \Lambda = 1$ (because the 2-form of curvature vanishes automatically) and in the Lagrangian case $\dim \Lambda = \dim \mathscr{M}$ (because $\tau = \{0\}$). That is another reason why the intermediate case $1 < \dim \Lambda < \dim \mathscr{M}$ is especially difficult for construction of quasimodes.

Below we investigate the *equations of zero curvature* for an adapted symplectic connection on a normal bundle over an isotropic submanifold Λ. We present the solutions of these equations in *adiabatic approximation* by using the symplectic version of the Berry connection (we consider only the simplest case when the eigenvalues of the variation matrix are of multiplicity 1; in more general cases the analogs of procedure [90] should be applied). Moreover, in general nonadiabatic case, we give the sufficient conditions of solvability for these equations and write the *explicit formulas for the solution in terms of infinitesimal symmetries* of the Hamilton flow over Λ. These formulas appear as a generalization of the Weinstein construction [87] for the Bott-type connection.

The structure of the text is the following. In Section 2 we introduce all basic geometrical objects over an isotropic submanifold Λ. In Section 3 we consider the quasimodes of type (1.2), (1.2a), where $\tilde{\varphi}(\alpha, y)$ does not depend on y. And we discuss here the analytical and topological possibilities to satisfy the quantization rule over Λ. In Section 4 the general case, where $\tilde{\varphi}(\alpha, y)$ is a polynomial in y, is considered. In Section 5 we show how an isotropic manifold Λ and invariant objects over it can appear from the action of a Lie group or a pseudogroup; in particular, from the Poisson action of a Poisson group [21, 33, 77]. The last Section 6 contains the formulas describing the connections with zero curvature. There are also three Appendices: about an integral cohomology class for non-Lagrangian submanifolds, about the first variation equation over an isotropic submanifold, and with the technical proof of the main commutation formula.

2. GEOMETRIC CONSTRUCTIONS

(i) *Basic Quartet*

Let $\mathscr{M}$ be a complete Riemann manifold and $\mathfrak{X} = T^*\mathscr{M}$. We start from the following objects:

(I) a compact connected isotropic submanifold $\Lambda \subset \mathfrak{X}$;

(II) a smooth positive measure $d\sigma$ over Λ normalized as follows $\int_\Lambda d\sigma = 1$;

(III) a normal positive subbundle τ over Λ;

(IV) a Hermitian connection ∇ over τ.

Below in this paragraph we shall explain the terminology used in (III), (IV) and determine certain geometric structures over Λ.

(ii) *Normal Positive Subbundle*

The phase space $\mathfrak{X} = T^*\mathscr{M}$ possesses a standard symplectic structure $\omega = dp \wedge dq$, where $q \in \mathscr{M}$, $p \in T^*_q\mathscr{M}$. At each point $(p, q) \in \mathfrak{X}$ there is a "vertical" plane $\mathscr{P}$ induced by the fields $\partial/\partial p_j$ and a "horizontal" plane Q induced by the fields $\partial/\partial q^j$. In other words, Q is the horizontal subspace for a linear connection on $T^*\mathscr{M}$ dual to the Levi-Civita connection on $T\mathscr{M}$. The bundles $\mathscr{P}$ and Q possess a natural Euclidean structure.

We have an isotropic submanifold Λ in $\mathfrak{X}$.

Denote by $T_\Lambda\mathfrak{X}$ a bundle over the base Λ with fibers $T_\alpha\mathfrak{X}$, $\alpha \in \Lambda$. The bundles $\mathscr{P}$ and Q are also restricted to the base Λ, but we shall not introduce the new notation $\mathscr{P}_\Lambda$ and Q_Λ in this case. All the bundles considered below have the base Λ.

By $\cdots^{\curlyvee}$ we shall denote the operation of *skew orthogonal complement* with respect to the form ω. A subbundle $\mathfrak{m} \subset T_\Lambda\mathfrak{X}$ is called *isotropic* (or *coisotropic*, *Lagrangian*) if $\mathfrak{m}^{\curlyvee} \supset \mathfrak{m}$ (or $\mathfrak{m}^{\curlyvee} \subset \mathfrak{m}$, $\mathfrak{m}^{\curlyvee} = \mathfrak{m}$).

Since Λ is an isotropic submanifold, we have $T\Lambda \subset (T\Lambda)^{\curlyvee}$. Thus the quotient bundle and the projection

$$\mathring{E} \stackrel{\text{def}}{=} (T\Lambda)^{\curlyvee}/T\Lambda, \qquad T\Lambda^{\curlyvee} \xrightarrow{\circ} \mathring{E}. \tag{2.0}$$

are defined. The fibres of $\mathring{E}$ are of dimension $\dim \mathfrak{X} - 2\dim \Lambda$, and have the symplectic structure induced by the form ω. The bundle $\mathring{E}$ is called a *normal symplectic bundle over* Λ (see [28, 87]).

We shall denote by ${}^{\mathbf{C}}(\cdots)$ the operation of complexification, and by $\overline{\cdots}$ the operation of complex conjugation in ${}^{\mathbf{C}}T_\Lambda\mathfrak{X}$. Introduce the following quadratic forms

$$(X, Y)_\pm = \pm\frac{i}{2}\,\omega(X, \bar{Y}). \tag{2.1}$$

A subbundle $\mathfrak{m} \subset {}^{\mathbf{C}}T_\Lambda\mathfrak{X}$ is called *positive* if the form $(..., ...)_+$ is positively defined on it, i.e., if this form defines a Hermitian structure on $\mathfrak{m}$. A *negative* subbundle is defined similarly by the norm $(..., ...)_-$. A positive Lagrangian subbundle will be called *Kählerian*, and a negative Lagrangian subbundle will be called *anti-Kählerian.*

The forms (2.1) are correctly projected on ${}^{\mathbf{C}}\mathring{E}$ by (2.0). On Kählerian subbundles $\tau \subset {}^{\mathbf{C}}\mathring{E}$ the form $(..., ...)_+$ defines a Hermitian structure. Such subbundles τ will be called *normal positive* over Λ.

(iii) *Almost Polarization*

Introduce an arbitrary Euclidean structure in $(T\Lambda)^Y$ and denote by E the orthogonal complement to $T\Lambda$ with respect to this structure. Then E is a symplectic subbundle in $T_\Lambda\mathfrak{X}$, and

$$E\subset(T\Lambda)^Y, \qquad \dim E=\dim\mathfrak{X}-2\dim\Lambda$$

The projection on the quotient

$$E\xrightarrow{\circ}\mathring{E} \qquad \text{or} \qquad {}^{\mathbf{C}}E\xrightarrow{\circ}{}^{\mathbf{C}}\mathring{E} \tag{2.2}$$

is an isomorphism. Here and below the considered morphisms of bundles are identical on the base Λ; another situation when the points of the base are moving, will be always pointed out especially.

By $\rho\subset{}^{\mathbf{C}}E$, we denote the pre-image of the bundle $\bar{\tau}$ under the projection (2.2). Then ρ is a negative isotropic subbundle in ${}^{\mathbf{C}}T_\Lambda\mathfrak{X}$, such that

$$\rho\subset({}^{\mathbf{C}}T\Lambda)^Y, \qquad {}^{\mathbf{C}}\dim\rho=\tfrac{1}{2}\dim\mathfrak{X}-\dim\Lambda, \tag{2.3}$$

and the projection of ρ on ${}^{\mathbf{C}}\mathring{E}$ coincides with $\bar{\tau}$.

Now consider the skew-orthogonal symplectic subbundle E^Y. Since $T\Lambda$ is a Lagrangian subbundle in E^Y, then an anti-Kählerian subbundle $\pi\subset{}^{\mathbf{C}}E^Y$ exists (see Lemma A.1 in Appendix A). In the whole bundle $T_\Lambda\mathfrak{X}$, the subbundle π is isotropic, negative, and moreover

$${}^{\mathbf{C}}T\Lambda\subset\pi\oplus\bar{\pi}, \qquad {}^{\mathbf{C}}\dim\pi=\dim\Lambda. \tag{2.4}$$

Denote by

$$\Pi=\rho\oplus\pi \tag{2.5}$$

the anti-Kählerian subbundle in ${}^{\mathbf{C}}T_\Lambda\mathfrak{X}$. It will be called an *almost polarization* over Λ. The word "almost" is used here, since there are no assumptions that Π is integrable (we recall that an integrable distribution of Lagrangian planes over $\mathfrak{X}$ is called a polarization [29, 54]).

(iv) *Elements of Volume and Measure of Nonunimodularity*

Lemma 2.1. *The subbundle π generates an Euclidean structure on $T\Lambda$ (or a Riemannian metrics on Λ) by the formula*

$$(u,v)_{T\Lambda}\overset{\text{def}}{=}(\mathrm{f}(u),\mathrm{f}(v))_-, \tag{2.6}$$

where f *is the projection* f: ${}^{\mathbf{C}}T\Lambda \to \pi$ *along* $\bar{\pi}$. *Conversely, for any smooth metric on* Λ, *a subbundle* π *exists with properties* (2.4) *such that the projection* f *is an isometry.*

Denote by $d\sigma^\pi$ the measure on Λ generated by the Euclidean structure (2.6) and consider the ratio of two elements of volume on Λ:

$$\Delta_\Lambda = d\sigma / d\sigma^\pi. \tag{2.7}$$

We shall see below (Lemma 5.1) that the function Δ_Λ is an *analog of the group modular function.*

Further, there is a horizontal lift

$$T_{q(\alpha)}\mathscr{M} \to Q(\alpha) \qquad \text{along} \quad \mathscr{P}(\alpha)$$

for all the points $\alpha \in \Lambda$ (here $q(\alpha)$ is the projection of α on $\mathscr{M}$). We shall identify $T_{q(\alpha)}\mathscr{M} \approx Q(\alpha)$.

Consider the projection

$$T_q\mathscr{M} \approx Q \xrightarrow{F} \Pi \qquad \text{along} \quad {}^{\mathbf{C}}\mathscr{P}. \tag{2.8}$$

Then $\mathscr{E}(\alpha) \overset{\text{def}}{=} F_\alpha(T_{q(\alpha)}\mathscr{M})$ is an Euclidean subspace in $\Pi(\alpha)$ with respect to the form $(..., ...)_-$. On the other hand the Riemann tensor given on $T\mathscr{M}$ is transported to the subbundle $\mathscr{E}$ by means of F. Thus we obtain a new Euclidean structure on $\mathscr{E}$. These two structures generate two elements of volume in the fibers of $\mathscr{E}$. Their ratio (of a new element to an old element) is a function on Λ. We denote this ratio by $\Delta_{\mathscr{M}}$. An explicit formula is the following

$$\Delta_{\mathscr{M}} = \frac{\det\, g(q(\alpha))^{1/2}}{\det\, \operatorname{Im} A(\alpha)^{1/2}}.$$

Here $g = ((g_{ij}))$ is the Riemann tensor on $\mathscr{M}$ in local coordinates $q = (q^1, ..., q^d)$, and

$$A = \omega\left(\frac{\partial}{\partial p} \overset{\otimes}{,} Z(\alpha)\right)^{-1} \cdot \omega\left(Z(\alpha) \overset{\otimes}{,} \frac{\partial}{\partial q}\right)\bigg|_{q = q(\alpha),\ p = p(\alpha)},$$

where $Z = \{Z_s\}$ is an arbitrary basis in Π (see also Appendix C).

Now we define

$$\Delta \overset{\text{def}}{=} \Delta_\Lambda \cdot \Delta_{\mathscr{M}}. \tag{2.9}$$

Note that $\Delta > 0$ uniformly on Λ. The difference between the function Δ and 1 can be called a *measure of nonunimodularity* of the almost polarization Π

with respect to Λ and $\mathscr{M}$. There is a close analogy with the measure of non-unimodularity in BRST-quantization [23, 57].

(v) *Gaussian Packets*

Let the isotropic submanifold $\Lambda \subset T^*\mathscr{M}$ be determined by equations

$$\Lambda = \{q = q(\alpha),\ p = p(\alpha)\}.$$

For each $\alpha \in \Lambda$ the Riemannian connection on $\mathscr{M}$ generates the exponential mapping $\exp_\alpha\colon T_{q(\alpha)}\mathscr{M} \to \mathscr{M}$. If a point q lies in a small neighborhood of the point $q(\alpha)$, then the vector

$$x(\alpha, q) \stackrel{\text{def}}{=} \exp_\alpha^{-1}(q) \in T_{q(\alpha)}\mathscr{M}. \tag{2.10}$$

is correctly defined. Its horizontal lift into $Q(\alpha)$ will be denoted by the same letter: $x(\alpha, q) \in Q(\alpha)$.

Let introduce a function

$$\begin{aligned}\Phi(\alpha, \alpha_0 \mid q) = \int_{\alpha_0}^{\alpha} p(\alpha)\cdot dq(\alpha) + p(\alpha)\cdot x(\alpha, q)\\ + \tfrac{1}{2}\omega(Fx(\alpha, q), x(\alpha, q)),\end{aligned} \tag{2.11}$$

where the projection (2.8) is used in the last summand. The function $\Phi(\alpha, \alpha_0 \mid q)$ is defined for all q close to $q(\alpha)$ (the local explicit expression is given in Appendix C).

We stress that

$$\operatorname{Im}\Phi \geqslant 0, \qquad \text{and} \qquad \operatorname{Im}\Phi = 0 \qquad \text{only if} \quad q = q(\alpha).$$

Thus for any $\alpha, \alpha_0 \in \Lambda$, the function

$$▯(\alpha, \alpha_0 \mid q) \stackrel{\text{def}}{=} \exp\left\{\frac{i}{h}\Phi(\alpha, \alpha_0 \mid q)\right\} \tag{2.12}$$

is defined mod $O(h^\infty)$ globally for all $q \in \mathscr{M}$ (by extending it to be zero outside a neighborhood of the set $\{q = q(\alpha)\}$; see Section 3(i) below). We shall call ▯ a *Gaussian packet* over Λ. This object depends only on Λ and on the almost polarization Π. Note that in the variable α both the function Φ and the Gaussian packet ▯ are defined on the universal covering over Λ.

The properties of Gaussian packets ▯ are described in Lemma C.1 of Appendix C. They were used in [40, 43] in the case when Λ is Lagrangian, and they are *key properties*. This is a consequence of the well-known fact that Gaussian packets (or "coherent states of the Heisenberg group") form an overcompleted system. This fact allows to replace (with a certain accuracy) the differentiation $\partial/\partial q$ and the multiplication by q of the function ▯ by the operators acting on ▯ in certain parameters. In contrast to the Lagrangian case, not only the coordinates $\alpha \in \Lambda$, but also the coordinates $\bar{z} \in \tau$ will now play the role of these parameters.

(vi) *Complex Quasi-orientation*

To each Hermitian bundle $\mathfrak{m}$ one can associate the $\mathbf{U}(1)$-bundle called *determinant* (or briefly det-). The transition functions λ_{IJ} for this new bundle have the form $\lambda_{IJ} = d_{IJ}/|d_{IJ}|$, where d_{IJ} are the determinants of transition matrices (from chart I to chart J) of the bundle $\mathfrak{m}$. We shall call $\mathfrak{m}$ *quasi-orientable* if there exists a local trivialization ε of $\mathfrak{m}$ for which all λ_{IJ} are constant (independent of a point at the intersection of charts). We shall call ε a *quasi-orientation* of the Hermitian bundle $\mathfrak{m}$.

Further, ε will be called a *pre-orientation* if all λ_{IJ} are equal to ± 1. If they all are equal to $+1$, we shall use the term *orientation*.

The cocycle defined by λ_{IJ} generates a cohomology class

$$\lambda^{\varepsilon} \in H^1(\Lambda, \mathbf{U}(1)) \qquad \text{or} \qquad \lambda^{\varepsilon} = \exp\{-i\pi w_1^{\varepsilon}\}, \qquad \text{where} \quad w_1^{\varepsilon} \in H^1(\Lambda, \mathbf{R}_2)$$

(we denote $\mathbf{R}_2 = \mathbf{R}/\mathrm{mod}\, 2$). Different quasi-orientations ε correspond one-to-one to different classes w_1^{ε}. If $w_1^{\varepsilon} \in H^1(\Lambda, \mathbf{Z}_2)$, then ε is a pre-orientation, and the class w_1^{ε} is an analog of the Stiefel–Whitney class.

Note that a Hermitian bundle $\mathfrak{m}$ over the base Λ is quasi-orientable iff its Chern class is trivial $c_1 = 0$ in the group $H^2(\Lambda, \mathbf{R})$ or iff a Hermitian connection with zero trace of curvature exists on $\mathfrak{m}$.

A Hermitian bundle is pre-orientable iff $2c_1 = 0$ in the group $H^2(\Lambda, \mathbf{Z})$, or iff $\det^2$-bundle is trivial. In this case the class w_1^{ε} is an obstruction to the trivialization of the det-bundle.

If an Euclidean subbundle exists in the given Hermitian $\mathfrak{m}$, then a natural pre-orientation exists (and in the case ${}^{\mathbf{C}}\dim \mathfrak{m} = 1$ the inverse statement also holds). We shall denote by the same letter ε the Euclidean subbundle and the pre-orientation generated by it. In this case the class w_1^{ε} coincides with the usual Stiefel–Whitney class w_1 of the subbundle ε.

And finally, a Hermitian bundle $\mathfrak{m}$ is orientable iff $c_1 = 0$ in the group $H^2(\Lambda, \mathbf{Z})$, or iff the det-bundle is trivial. Of course, the Stiefel–Whitney class: $w_1^{\varepsilon} = 0$ is trivial for any orientation ε.

(vii) *Integral Class and Topological Phase over Isotropic Submanifold*

LEMMA 2.2. *For any normal positive subbundle τ over an isotropic submanifold Λ the Chern class is zero*: $c_1 = 0$ *in* $H^2(\Lambda, \mathbf{Z})$, *i.e.,* τ *is quasi-orientable.*

Let us fix a certain quasi-orientation ε in τ. This quasi-orientation locally determines an Euclidean subbundle $\varepsilon_{\mathscr{U}} \subset \tau|_{\mathscr{U}}$ in each chart $\mathscr{U} \in \Lambda$. Its real part $l_{\mathscr{U}} = \mathrm{Re}(\varepsilon_{\mathscr{U}})$ is a Lagrangian subbundle in $\mathring{E}|_{\mathscr{U}}$ (see Appendix A). The pre-image of $l_{\mathscr{U}}$ under the projection (2.0) will be denoted by $L_{\mathscr{U}}$. Then $L_{\mathscr{U}}$ is Lagrangian in $T_\lambda \mathfrak{X}$.

So, we have two Lagrangian subbundles $L_{\mathscr{U}}$ and $\mathscr{P}$ over the chart $\mathscr{U}$, and a fixed anti-Kählerian subbundle Π. Hence, we can define a differential 1-form of the Arnold type. It is defined in the chart $\mathscr{U}$ by the formula

$$\mu^\varepsilon \stackrel{\text{def}}{=} \mu(L_{\mathscr{U}}, \mathscr{P})$$

(see Definition (A.1) in Appendix A).

Lemma 2.3. *The real closed form μ^ε is globally defined on an isotropic submanifold Λ. If the quasi-orientation ε is a pre-orientation, then the cohomology class of the form μ^ε is integral: $[\mu^\varepsilon] \in H^1(\Lambda, \mathbf{Z})$, does not depend on ε, and*

$$[\mu^\varepsilon] = w_1^\varepsilon + w_1(\Lambda) + w_1(\mathscr{M}) \pmod 2$$

Corollary 2.1. *If the doubled Chern class $2c_1$ of a normal positive subbundle τ over an isotropic submanifold Λ is zero in $H^2(\Lambda, \mathbb{Z})$, then τ generates an integral class from $H^1(\Lambda, \mathbb{Z})$ represented by Arnold's form. In the case of Lagrangian Λ* (*i.e., when $\tau = \{0\}$*), *the definition of Arnold's form coincides with* (1.4), *and its class coincides with the Maslov class.*

The integral

$$-\tfrac{1}{2}\int_{\alpha_0}^{\alpha} \mu^\varepsilon$$

will be called a *topological phase* over Λ. This phase depends on the quasi-orientation ε of the normal positive subbundle τ over Λ.

The appearance of the term "phase" will be explained below.

(viii) *Gauge Phase and Liouville Equation*

Denote by $\theta = \{\theta_{\mathscr{U}}\}$ the matrices of the form of connection ∇ calculated in local charts $\mathscr{U} \subset \Lambda$ with respect to arbitrary bases in Euclidean subbundles $\varepsilon_{\mathscr{U}}$ (see subsection (vii)). The imaginary part of the trace $\mathrm{tr}_\varepsilon\, \theta$ is independent of the choice of bases and defines a global complex 1-form over Λ. Denote

$$\beta^\varepsilon \stackrel{\text{def}}{=} \mathrm{Im}(\mathrm{tr}_\varepsilon\, \theta)$$

The differential of the 1-form $i\beta^\varepsilon$, obviously, coincides with the trace of the form of curvature.

LEMMA 2.4. *On an isotropic submanifold Λ the real* 1*-form*

$$\varkappa = \beta^\varepsilon + \pi\mu^\varepsilon \tag{2.13}$$

does not depend on the quasi-orientation ε of a normal positive subbundle τ. The differential $id\varkappa$ coincides with the trace of curvature form for the Hermitian connection on τ.

We call the form $\varkappa$ a *Gauge form* of the normal positive subbundle τ over Λ. If the trace of curvature on τ is equal to zero, then the integral $-\frac{1}{2}\int_{\alpha_0}^{\alpha}\varkappa$ will be called a *Gauge phase*.

Now we explain the appearance of the term "phase".

Suppose $H = H(q, p)$ is a certain smooth real function on $\mathfrak{X}$. We shall say that H is *compatible* with $(\Lambda, \sigma, \tau, \nabla)$ if the following conditions hold:

(i) Λ and $d\sigma$ are invariant with respect to the flow of the Hamiltonian field $\operatorname{ad}(H)$,

(ii) τ is invariant with respect to the projection of the first variation equation of the field $\operatorname{ad}(H)$ on ${}^{\mathbf{C}}\mathring{E}$ and this projection is *adaptable* to the connection ∇.

The latter condition means that the translation by the variation equation in τ coincides with the parallel transport along the trajectory of the field $\operatorname{ad}(H)$ with respect to the connection ∇ (see Appendix B for details).

By ${}^{\mathscr{E}}\operatorname{Var}(H)$ we shall denote the matrix of projection of the first variation equation on Π, calculated with respect to the basis in the Euclidean subbundle $\mathscr{E} \subset \Pi$ (see subsection (iv) and Appendix B).

LEMMA 2.5. (a) *The measure of nonunimodularity Δ satisfies the equation*

$$\operatorname{ad}(H) \ln \Delta = \operatorname{Re} \operatorname{tr}({}^{\mathscr{E}}\operatorname{Var}(H)).$$

(b) *Suppose the function H is compatible with $(\Lambda, \sigma, \tau, \nabla)$ and the trace of curvature of the connection ∇ is equal to zero. Then the function*

$$J_0(\alpha) = \frac{1}{\Delta(\alpha)} \exp\left\{-i\int_{\alpha_0}^{\alpha}\varkappa\right\} \tag{2.14}$$

is a solution of the following Liouville equation on the universal covering of an isotropic submanifold Λ:

$$\operatorname{ad}(H)\, J_0 + \operatorname{tr}({}^{\mathscr{E}}\operatorname{Var}(H))\, J_0 = 0. \tag{2.15}$$

By the definition of the form $\varkappa$, we have

$$\sqrt{J_0(\alpha)} = \frac{1}{\sqrt{\Delta(\alpha)}} \exp\left\{-\frac{i}{2}\int_{\alpha_0}^{\alpha} \beta^{\varepsilon} - i\frac{\pi}{2}\int_{\alpha_0}^{\alpha} \mu^{\varepsilon}\right\}. \tag{2.14a}$$

Thus the phase of the function $\sqrt{J_0}$ splits into two summands. The first summand is determined by the connection on τ. The second (topological) summand depends only on the geometry of $\mathscr{M}$, Λ, τ and on the quasi-orientation ε. In particular, if ε is a pre-orientation (for example, ε is an Euclidean subbundle in τ), the contribution of the second summand in the phase (2.14a), after moving around closed cycles on Λ, is a multiple of $\pi/2$. If $\mathscr{M}$, Λ, and τ are orientable, the topological contribution to the phase of the function $\sqrt{J_0}$, after moving around closed cycles, is a multiple of π.

(ix) *Phases of Excitations*

Assume that the following additional object exists:

(V) one-dimensional ∇-invariant Hermitian subbundle $\tau_1 \subset \tau$, on which the curvature vanishes.

At the initial point $\alpha_0 \in \Lambda$, we fix an arbitrary vector $V_{10} \in \tau_1(\alpha_0)$, and consider a quasi-orientation ε_1 in τ_1 containing this vector. We denote by $i\beta_1$ the form of connection with respect to this quasi-orientation. The real form β_1 can be calculated by the formula

$$\nabla \mathring{X}_1 = i\beta_1 \mathring{X}_1, \tag{2.16}$$

where $\mathring{X}_1$ is a local section of the quasi-orientation ε_1 with the unit norm.

We have another possible expression for β_1

$$i\beta_1 = (d\mathring{X}_1, \mathring{X}_1)_+ - \frac{(dV_1, \mathring{X}_1)_+}{(V_1, \mathring{X}_1)_+},$$

where V_1 is any local ∇-parallel section of the subbundle τ_1 (actually, the second summand in this formula is independent of the choice of the section V_1; the differentials $d\cdots$ are understood as differentials of coordinates of sections with respect to a local basis in τ_1).

Such a partition of the form β_1 into two summands, "geometrical" and "dynamical", is similar to the Aharonov–Anandan procedure [1] or, in the adiabatic approximation, to the Berry procedure [10]. Of course, this partition exists also in abstract Hermitian bundles (see [13, 78]).

The integral $-\int_{\alpha_0}^{\alpha} \beta_1$ will be called a *phase of excitation* along τ_1.

3 VACCUM QUASIMODES

(i) *Intertwining Operator over Isotropic Submanifolds*

We introduce the space of finite smooth functions $\psi_\hbar$ on $\mathscr{M}$ smoothly depending on a parameter $\hbar \in (0, 1]$ and such that all the norms

$$\|(\hbar\, \partial/\partial q)^k\, \psi_\hbar\|_{L^2(\mathscr{M})} \qquad |k| = 0, 1, 2, \ldots \tag{3.1}$$

are uniformly bounded in $\hbar$. In this space, there is the subspace consisting of functions, whose norms are of order $O(\hbar^\infty)$. The quotient space will be denoted by $C^\infty_\hbar(\mathscr{M})$.

Below, expressions $O(\hbar^N)$ will denote elements from $C^\infty_\hbar(\mathscr{M})$ with estimate in the sense of norms (3.1) or the operators on $C^\infty_\hbar(\mathscr{M})$ with the same estimate.

Suppose $f = f(q, p)$ is a smooth function on $T^*\mathscr{M}$ increasing with all derivatives not greater than a polynomial as $|p| \to \infty$. Such functions will be called *symbols* on $T^*\mathscr{M}$. To each symbol we assign an operator

$$\hat{f}\colon C^\infty_\hbar(\mathscr{M}) \to C^\infty_\hbar(\mathscr{M}),$$

defined in local coordinates by the formula

$$\hat{f} = \det g(q)^{-1/4} \cdot f(q, -i\hbar\, \partial/\partial q) \cdot \det g(q)^{1/4} + O(\hbar^2). \tag{3.2}$$

Here $g = ((g_{ij}))$ is the Riemann tensor on $\mathscr{M} = \mathscr{M}^d$ in local coordinates $q = (q^1, \ldots, q^d)$; the operator $f(q, -i\hbar\, \partial/\partial q)$ is defined in the Weyl sense; the corrections $O(\hbar^2)$ provide the global matching of local expressions (3.2) to an operator over $\mathscr{M}$ (for details about the operators over manifolds, see in [71, 82]). We shall not be interested in the explicit form of the correction $O(\hbar^2)$, since it is beyond the accuracy of asymptotics constructed below.

The standard example of symbols is: $f(q, p) = \sum_{jk=1}^d g^{jk}(q)\, p_j p_k$. In this case formula (3.2) yields $\hat{f} = -\hbar^2 \Delta + O(\hbar^2)$, where Δ is the Laplace–Beltrami operator, and the correction $O(\hbar^2)$ is calculated in terms of the curvature of the Riemannian connection. Note that $\hbar^2 \Delta = O(1)$ in the sense of norms (3.1), but $\hbar^2 \Delta \neq O(\hbar^2)$.

Now we consider the Gaussian packets $\mathbb{I}(\alpha, \alpha_0 \mid q) = \exp\{(i/\hbar)\, \Phi(\alpha, \alpha_0 \mid q)\}$ from (2.2). Obviously, for each $\alpha, \alpha_0 \in \Lambda$, the function $\mathbb{I}(\alpha, \alpha_0 \mid \cdot)$ defines a element form $C^\infty_\hbar(\mathscr{M})$.

Denote by $\mathscr{F}(\Lambda) \equiv C^\infty(\Lambda)$ the space of all smooth complex-valued functions on Λ, and denote by $\mathscr{F}_0(\Lambda)$ the *subspace of functions whose supports are contained in simply connected neighborhoods of the fixed point* $\alpha_0 \in \Lambda$.

Following [40], define an integral operator $\mathbb{I}_\Lambda\colon \mathcal{F}(\Lambda) \to C_\hbar^\infty(\mathcal{M})$,

$$\mathbb{I}_\Lambda(\varphi)(q) \stackrel{\text{def}}{=} \frac{1}{c} \int_\Lambda \varphi(\alpha)\, \mathbb{I}(\alpha, \alpha_0 \mid q) \frac{d\sigma(\alpha)}{\sqrt{\Delta(\alpha)}}, \tag{3.3}$$

where Δ is the measure of nonunimodularity (2.9) and

$$c = 4^{\dim \Lambda/4} \cdot (\pi\hbar)^{(\dim \Lambda + \dim \mathcal{M})/4}. \tag{3.4}$$

LEMMA 3.1. *For any symbol f on $T^*\mathcal{M}$ and any $\varphi \in \mathcal{F}(\Lambda)$, we have the following formula for asymptotics of average*

$$(\hat{f}\,\mathbb{I}_\Lambda(\varphi), \mathbb{I}_\Lambda(\varphi))_{L^2(\mathcal{M})} = \int_\Lambda f(q(\alpha), p(\alpha))\, |\varphi(\alpha)|^2\, d\sigma(\alpha) + O(\hbar).$$

If the symbol f vanishes in a neighborhood of Λ, then the average written above is of order $O(\hbar^\infty)$.

Thus the *oscillation front* (or the frequency set, see [26, 46]) of the wave function

$$\psi_\hbar(q) \stackrel{\text{def}}{=} \mathbb{I}_\Lambda(\varphi)(q)$$

coincides with supp φ on the isotropic submanifold $\Lambda \subset T^*\mathcal{M}$.

We have the following "isotropic" analog of the main theorem proved in [40] for the case of Lagrangian Λ.

THEOREM 3.1. *Suppose a real symbol H is compatible with the following quartet:*

(I) *an isotropic submanifold Λ,*

(II) *a measure $d\sigma$ on Λ,*

(III) *a normal positive subbundle τ over Λ,*

(IV) *a Hermitian connection ∇ on τ.*

Then the commutation formula

$$\hat{H} \cdot \mathbb{I}_\Lambda = \mathbb{I}_\Lambda \cdot \left(H - i\hbar\, \mathrm{ad}(H) + \frac{\hbar}{2} \langle \varkappa, \mathrm{ad}(H) \rangle \right) + O(\hbar^{3/2}) \tag{3.5}$$

holds on the subspace $\mathcal{F}_0(\Lambda)$. In the right-hand side of this formula the symbol H and the field ad(H) *are restricted to Λ, and $\varkappa$ is a gauge form of the subbundle τ.*

The scheme of the proof is given in the Appendix C.

The operator in the right-hand side of (3.5) can be simplified under the condition $d\varkappa = 0$.

COROLLARY 3.1. *Suppose, in addition to Theorem* 3.1, *the trace of curvature for the connection* ∇ *is equal to zero; then*

$$\hat{H} \cdot \mathbb{I}_{\Lambda} \cdot \exp\left\{-i\int_{\alpha_0}^{\alpha} \varkappa/2\right\}$$
$$= \mathbb{I}_{\Lambda} \cdot \exp\left\{-i\int_{\alpha_0}^{\alpha} \varkappa/2\right\} \cdot (H - i\hbar\, \mathrm{ad}(H)) + O(\hbar^{3/2})$$

on the subspace $\mathscr{F}_0(\Lambda)$. *This commutation formula holds on the whole space* $\mathscr{F}(\Lambda)$ *iff*

$$\frac{1}{2\pi}\left[\hbar^{-1} p\, dq|_{\Lambda} - \varkappa/2\right] \in H^1(\Lambda, \mathbf{Z}). \tag{3.6}$$

Here the square brackets denote the cohomology class corresponding to the differential form.

Condition (3.6) is called the *quantization rule*. We see that the Gauge form of a normal positive subbundle over Λ plays here the role of a correction to the Poincaré–Cartan form. The general definition of the cohomology class $[\varkappa]$ in the quantization rule over an isotropic Λ was not known earlier, but we stress that for partial situations this class (not a form $\varkappa$) in other terminology appeared in the works [7, 16, 31, 75] for the case $\dim \Lambda = 1$ and in [8, 67] (for the case $1 \leqslant \dim \Lambda < \dim \mathscr{M}$). This remark also concerns the quantization rule (4.17) below.

COROLLARY 3.2. *Suppose the symbol* H *is compatible with* $(\Lambda, \sigma, \tau, \nabla)$, *the trace of curvature for* ∇ *is equal to zero and* $H|_{\Lambda} \equiv \lambda = const$. *Suppose also that the function* $\mathscr{S} \in \mathscr{F}(\Lambda)$ *is an eigenfunction for the Hamilton field*

$$\mathrm{ad}(H)|_{\Lambda}\, \mathscr{S} = i\lambda_1 \mathscr{S}, \qquad \lambda_1 = const, \tag{3.7}$$

and suppose the quantization rule (3.6) *holds. Then the function*

$$\psi = \mathbb{I}_{\Lambda}\left(\exp\left\{-i\int \varkappa/2\right\} \mathscr{S}\right),$$

or more precisely,

$$\psi(q)=\frac{1}{c}\int_{\Lambda}\mathscr{S}(\alpha)\exp\left\{\frac{i}{h}\Phi(\alpha,\alpha_0\,|\,q)-i\int_{\alpha_0}^{\alpha}\varkappa/2\right\}\frac{d\sigma(\alpha)}{\sqrt{\Delta(\alpha)}},\tag{3.8}$$

on the Riemann manifold $\mathscr{M}$ *satisfies the estimates*

$$\begin{aligned}\|\hat{H}\psi-(\lambda+h\lambda_1)\,\psi\|_{L^2(\mathscr{M})}&=O(h^{3/2}),\\ \|\psi\|_{L^2(\mathscr{M})}&=1+O(h).\end{aligned}\tag{3.9}$$

So, if the operator $\hat{H}$ *is essentially selfadjoint in* $L^2(\mathscr{M})$, *then the distance from the point* $\lambda+h\lambda_1$ *to the exact spectrum of operator* $\hat{H}$ *is of order* $O(h^{3/2})$.

The function ψ satisfying estimates (3.9) is called, by terminology of [4], a *quasimode* of the operator $\hat{H}$ corresponding to the *quasienergy* $\lambda+h\lambda_1$.

Note that the statements of Corollaries 3.1 and 3.2 are not absolutely correct in the given form. Actually, if the parameter h tends to zero in an arbitrary way and if the submanifold Λ is fixed, the left-hand side of (3.6) cannot take only integer values, as the quantization rule (3.6) requires.

So, in Corollaries 3.1 and 3.2, we must suppose that one of the two following statements holds: either (A) h tends to zero along a discrete set of values, for which the rule (3.6) is satisfied, or (B) there is a family of submanifolds Λ, as well as of subbundles τ and connections ∇ over them (this is typical, for instance, for integrable systems), and the quantization rule (3.6) separates a discrete subfamily from this whole family.

For simplicity, we do not include these details in the statements of Corollaries 3.1 and 3.2. But we stress that assumption (B) is not satisfied in important examples, where the submanifold Λ is *isolated*. Now we shall show how one can avoid these difficulties when the structure of the field $\mathrm{ad}(H)$ on Λ is "good" enough.

(ii) *Resolution of Quantization Rule*

We recall that in the general case [25] the homology group splits into the sum

$$H_1(\Lambda)=\mathrm{Tors}(H_1(\Lambda))\oplus\tilde{H}_1(\Lambda).$$

Here Tors means a torsion of an abelian group; $\tilde{H}_1$ is the group without torsion dual to the cohomology group $H^1(\Lambda,\mathbf{R})$. Thus, closed forms $\{\zeta^j\}$

exists on Λ such that their classes $[\zeta^j]$ define a basis in $H^1(\Lambda, \mathbf{R})$ dual to a basis of cycles $\Gamma_i \in \tilde{H}_1(\Lambda)$

$$\frac{1}{2\pi} \oint_{\Gamma_i} \zeta^j = \delta_i^j, \qquad i, j = 1, ..., k' \equiv \dim H^1(\Lambda, \mathbf{R}). \tag{3.10}$$

We shall say that a vector field u over Λ is *almost periodic* if there is a basis of forms $\{\zeta^j\}$ with property (3.10) and such that

$$\langle \zeta^j, u \rangle = \mathrm{const}^j, \qquad j = 1, ..., k'.$$

The constants const^j will be called *frequencies* of the field u. A field u will be called *nonresonant* if the equation

$$u(\mathscr{S}) = \mathscr{N} \tag{3.11}$$

is solvable in the space of functions $\mathscr{S} \in \mathscr{F}(\Lambda)$ for any right-hand side $\mathscr{N} \in \mathscr{F}(\Lambda)$ with zero average $\int_\Lambda \mathscr{N}(\alpha)\, d\sigma(\alpha) = 0$; (here $d\sigma$ is a certain positive u-invariant measure on Λ).

Of course, if Λ is a torus $\mathbf{T}^k$ with angle coordinates $\alpha^1, ..., \alpha^k$, then $k' = k$, $\zeta^j = d\alpha^j$, and an almost periodic field has the form $u = \sum \mathrm{const}^j\, \partial/\partial\alpha^j$. Such a field will be nonresonant if its frequencies are arithmetically independent in the sense of KAM-theory [6, 59].

In the general case we note the following simple fact.

LEMMA 3.2. *Any nonresonant field on Λ is almost periodic.*

And now return to (3.6). Introduce the numbers

$$\delta_j(\hbar) = \frac{1}{2\pi h} \oint_{\Gamma_j} p\, dq - \left[\left[\frac{1}{2\pi h} \oint_{\Gamma_j} p\, dq \right]\right]. \tag{3.12}$$

Here $\Gamma_j \in \tilde{H}_1(\Lambda)$, and by double brackets we denote the integer part of a number. Obviously, $0 \leqslant \delta_j(\hbar) < 1$.

Also fix an arbitrary set of integer numbers $m = (m_1, ..., m_{k'})$. We define

$$\varkappa \overset{\text{def}}{=} \frac{1}{2\pi} \oint_{\Gamma_j} \varkappa \tag{3.13}$$

and introduce the function

$$\varphi_m(\alpha) = \exp\left\{ i \int_{\alpha_0}^{\alpha} \left(\sum_{j=1}^{k'} (m_j - \delta_j(\hbar) + \varkappa_j/2)\, \zeta^j - \varkappa/2 \right) \right\}. \tag{3.14}$$

on the universal covering of Λ. This function satisfies the equation

$$(-i\,\mathrm{ad}(H)+\tfrac{1}{2}\langle \varkappa, \mathrm{ad}(H)|_{\Lambda}\rangle)\,\varphi_m=\left(\sum_{j=1}^{k'}(m_j-\delta_j(\hbar)+\varkappa_j/2)\,c^j\right)\varphi_m,$$

where

$$c^j \stackrel{\mathrm{def}}{=} \langle \zeta^j, \mathrm{ad}(H)|_{\Lambda}\rangle. \tag{3.15}$$

If we substitute such function $\varphi=\varphi_m$ into (3.3), the quantization rule must be deformed as follows:

$$\frac{1}{2\pi}\left[\hbar^{-1}p\,dq|_{\Lambda}-\varkappa/2+\sum_{j=1}^{k'}(m_j-\delta_j(\hbar)+\varkappa_j/2)\,\zeta^j\right]\in H^1(\Lambda,\mathbf{Z}).$$

The definition of the coefficients $\delta_j(\hbar)$, $\varkappa_j$ implies that this deformed quantization rule holds automatically.

COROLLARY 3.2-a. *Suppose the quartet $(\Lambda,\sigma,\tau,\nabla)$ is compatible with the symbol H, the trace of curvature of connection ∇ is equal to zero. Moreover, suppose $H|_{\Lambda}=\lambda=\mathrm{const}$ and the field $\mathrm{ad}(H)|_{\Lambda}$ is almost periodic. Then the function $\psi_m=\mathbb{I}_{\Lambda}(\varphi_m)$, where φ_m is defined in (3.14), is a quasimode of the operator $\hat{H}$ related to the quasienergy $\lambda_m=\lambda+\hbar\lambda_{1m}$. Here*

$$\lambda_{1m}=\sum_{j=1}^{k'}(m_j-\delta_j(\hbar)+\varkappa_j/2)\,c^j,$$

and the constants $\delta_j(\hbar)$, $\varkappa_j$, and c^j are defined in (3.12), (3.13) and (3.15).

We can also formulate another version of this statement, without assuming that the trace of curvature is equal to zero, but imposing more rigid demands on the field $\mathrm{ad}(H)|_{\Lambda}$.

Suppose $\mathrm{ad}(H)|_{\Lambda}$ is a nonresonant field. Then a function $\mathscr{S}^{\varkappa}\in\mathscr{F}(\Lambda)$ exists such that

$$\mathrm{ad}(H)\,\mathscr{S}^{\varkappa}=c_{\varkappa}-\tfrac{1}{2}\langle\varkappa, \mathrm{ad}(H)|_{\Lambda}\rangle,$$

where

$$c_{\varkappa}\stackrel{\mathrm{def}}{=}\frac{1}{2}\int_{\Lambda}\langle\varkappa, \mathrm{ad}(H)|_{\Lambda}\rangle\,d\sigma. \tag{3.16}$$

We set

$$\varphi_m(\alpha) = \exp\left\{ i \int_{\alpha_0}^{\alpha} \sum_{j=1}^{k'} (m_j - \delta_j(\hbar))\, \zeta^j + i\mathscr{S}^{\varkappa}(\alpha) \right\}. \tag{3.17}$$

If we substitute the function $\varphi = \varphi_m$ into (3.3), then the quantization rule (3.6) varies as follows

$$\frac{1}{2\pi} \left[\hbar^{-1} p\, dq|_{\Lambda} + \sum_{j=1}^{k'} (m_j - \delta_j(\hbar))\, \zeta^j \right] \in H^1(\Lambda, \mathbf{Z}).$$

The definition of the coefficients $\delta_j(\hbar)$ implies that this rule holds automatically. This yields.

COROLLARY 3.2-b. *Suppose the quartet* $(\Lambda, \sigma, \tau, \nabla)$ *is compatible with symbol* H. *Suppose* $H|_{\Lambda} = \lambda = \mathrm{const}$ *and* $\mathrm{ad}(H)|_{\Lambda}$ *is a nonresonant field. Then the function* $\psi_m = \mathbb{I}_{\Lambda}(\varphi_m)$, *where* φ_m *is defined in* (3.17), *is a quasimode of the operator* $\hat{H}$ *related to the quasienergy* $\lambda_m = \lambda + \hbar\lambda_{1m}$. *here*

$$\lambda_{1m} = \sum_{j=1}^{k'} (m_j - \delta_j(\hbar))\, c^j + c_{\varkappa},$$

and the constants $\delta_j(\hbar)$, $c_{\varkappa}$ *are defined in* (3.12), (3.16).

Note that no quantization rule is mentioned in Corollaries 3.2-a and 3.2-b. In fact, by means of the condition of almost periodicity or of the nonresonant condition, we calculated explicitly an equidistant series of eigenvalues $\lambda + \hbar\lambda_{1m} + O(\hbar^{3/2})$ for the operator $\hat{H}$.

In this series the initial number $m = 0$ corresponds to the "*vacuum*" quasienergy

$$\lambda_0 \equiv \lambda + \hbar \left(c_{\varkappa} - \sum_{j=1}^{k'} \delta_j(\hbar)\, c^j \right) \tag{3.18}$$

the closest to the classical energy $\lambda = H|_{\Lambda}$. The corresponding "*vacuum*" *quasimode* has the form $\psi_0 = \mathbb{I}_{\Lambda}(\varphi_0)$. The quantum numbers $m_j = \pm 1, \pm 2, \ldots$ correspond to excitation of the state ψ_0 along he "action variables"; more precisely, these excitations are generated by the elements Γ_j from the subgroup without torsion $\tilde{H}_1(\Lambda) \subset H_1(\Lambda)$.

4. INTEGRAL REPRESENTATIONS FRAMED BY POLYNOMIALS

The picture of the quantum spectrum generated by an isotropic submanifold Λ obtained in the previous section can be made essentially more precise. Excitations are possible not only along "action variables", but also along subbundles in τ. Such excitations also have integral representations of the form (3.8), but are based on Gaussian packets framed by polynomials (such packets were used, for example, in [7, 67] and are very popular in the wavelet approximation). We shall see that integral representation for a quasimode has the form (1.2).

(i) *General Commutation Formula*

We need certain auxiliary notation. Consider the subbundle $\rho \subset \Pi$ introduced in subsection (iii) of Section 2. We recall that $\mathring{\rho} = \bar{\tau}$, where by the circle we denote the projection (2.0). We also denote by ρ^* the subbundle *dual* to ρ (the fibers $\rho^*(\alpha)$ are spaces of linear functionals on the fibers $\rho(\alpha)$; here $\alpha \in \Lambda$). The Hermitian structure defines a standard *isomorphism*,

$$\rho^* \to \bar{\rho}.$$

More precisely, to each $\xi \in \rho^*(\alpha)$ a vector $\bar{z} \in \bar{\rho}(\alpha)$ is assigned by the formula

$$\xi \cdot w \overset{\text{def}}{=} (w, z)_- \qquad \forall w \in \rho(\alpha). \tag{4.1}$$

In the similar way, we can identify $\bar{\tau}^* \to \tau$. And the same holds for complex-conjugate subbundles.

Now consider the projection

$$^{\rho}\mathrm{f}\colon \Pi \to \rho \qquad \text{along } \pi. \tag{4.2}$$

We have the sequence of morphisms (2.8), (4.2)

$$T_q \mathscr{M} \approx Q \xrightarrow{F} \Pi \xrightarrow{^{\rho}\mathrm{f}} \rho \xrightarrow{\circ} \bar{\tau}.$$

Consider the mapping

$$k \overset{\text{def}}{=} \circ \cdot {}^{\rho}\mathrm{f} \cdot F, \qquad k_\alpha\colon T_{q(\alpha)} \mathscr{M} \to \bar{\tau}(\alpha), \tag{4.3}$$

as well as the mapping of spaces of functions

$$k^*\colon \mathscr{F}(\bar{\tau}) \to \mathscr{F}(Q), \qquad (k^*\varphi)(\alpha, y) \overset{\text{def}}{=} \varphi(\alpha, k_\alpha(y)) \qquad \forall y \in Q(\alpha).$$

Here $\mathscr{F}(\bar{\tau})$ denotes a space of smooth functions on $\bar{\tau}$ which are *polynomial along the fibers*. One can regard $\mathscr{F}(\bar{\tau})$ as a space of sections $\Gamma^\infty(\mathrm{Pol}(\bar{\tau}))$, where $\mathrm{Pol}(\bar{\tau})$ is a fibration over Λ, whose fibers consist of polynomials on the fibers $\bar{\tau}$.

We also shall need below a subspace $\mathscr{F}_0(\bar{\tau}) \subset \mathscr{F}(\bar{\tau})$ consisting of functions whose supports on the base Λ lie in a simply connected domain.

Further, we introduce the projection

$$B\colon \bar{\rho} \to \rho \qquad \text{along} \quad {}^{\mathbf{C}}\mathscr{P} \oplus \pi.$$

This projection correctly defines the morphism of quotient bundles $B\colon \tau \to \bar{\tau}$. Taking into account the above identification (4.1) $\bar{\tau}^* \to \tau$, we obtain a fiber by fiber isomorphism $b\colon \bar{\tau}^* \to \bar{\tau}$ or a *symmetric bivector field* $b \in \Gamma^\infty(\bar{\tau} \otimes \bar{\tau})$. We associate to this field the second-order operator in $\mathscr{F}(\bar{\tau})$

$$\breve{b} \overset{\text{def}}{=} \frac{1}{4}\, b(\alpha)\, \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial z} \equiv \frac{1}{4} \sum_{\mu,\, \nu} b(\alpha)^{\mu\nu} \frac{\partial^2}{\partial z^\mu\, \partial z^\nu}, \tag{4.4}$$

where $\alpha \in \Lambda$, and by z^μ we denote the coordinates of a point in the fiber $\bar{\tau}(\alpha)$ with respect to the same basis, for which we calculate the matrix $b(\alpha)^{\mu\nu}$ of the mapping $b(\alpha)\colon \bar{\tau}(\alpha)^* \to \bar{\tau}(\alpha)$.

Now introduce the following integral operator $\square_\Lambda\colon \mathscr{F}(\bar{\tau}) \to C_h^\infty(\mathscr{M})$,

$$\square_\Lambda(\varphi)(q) \overset{\text{def}}{=} \frac{1}{c} \int_\Lambda \square(\alpha, \alpha_0 \mid q)\, \tilde{\varphi}\left(\alpha, \frac{x(\alpha, q)}{\sqrt{h}}\right) \frac{d\sigma(\alpha)}{\sqrt{\Delta(\alpha)}}. \tag{4.5}$$

Here

$$\tilde{\varphi} \overset{\text{def}}{=} k^* e^{-\breve{b}} \varphi, \qquad \varphi \in \mathscr{F}(\bar{\tau}),$$

where the mapping k was defined in (4.3), and the operator $\breve{b}$ in (4.4). The vector $x(\alpha, q) = \exp_\alpha^{-1}(q)$ was defined in (2.10), $\Delta(\alpha)$ is the measure of nonunimodularity (2.9), the constant c and the function $\square$ were defined in (3.4) and (2.12).

The Hamiltonian flow γ_H^t generated by the field $\operatorname{ad}(H)$ on $\mathscr{X}$ defines a flow on the bundle ${}^{\mathbf{C}}T_\Lambda \mathscr{X}$ (because Λ is an invariant submanifold). So, we also have a flow on the bundle $\bar{\tau}$. The generator of the last flow we denote by D_H (the explicit formula for D_H see in Appendix B, formula (B.2a)).

We have the following generalization of Theorem 3.1

THEOREM 4.1. *Suppose a symbol H on $T^*\mathscr{M}$ is compatible with $(\Lambda, \sigma, \tau, \nabla)$. Then*:

(a) *the commutation formula holds on $\mathscr{F}_0(\bar{\tau})$*

$$\hat{H} \cdot \square_\Lambda = \square_\Lambda \cdot \mathscr{L}(H) + O(h^{3/2}). \tag{4.6}$$

Here

$$\mathscr{L}(H) \overset{\text{def}}{=} H|_\Lambda - ih\, D_H + \frac{h}{2} \langle \varkappa, \operatorname{ad}(H)|_\Lambda \rangle, \tag{4.7}$$

where $\varkappa$ is the Gauge form of the normal subbundle τ over an isotropic Λ;

(b) *if the trace of curvature-form for the connection* ∇ *on* τ *is zero, then formula* (4.6) *is replaced by the following one*

$$\hat{H} \cdot \mathbb{I}_{\Lambda} \cdot e^{-(i/2) \int_{\alpha_0}^{\alpha} \varkappa} = \mathbb{I}_{\Lambda} \cdot e^{-i/2 \int_{\alpha_0}^{\alpha} \varkappa} \cdot (H|_{\Lambda} - i\hbar\, D_H) + O(\hbar^{3/2}). \tag{4.8}$$

This formula holds on the whole space $\mathscr{F}(\bar{\tau})$ *if the quantization rule* (3.6) *holds* (*together with all remarks in Section* 3 *concerning this variant*).

The proof of this theorem is given in Appendix C.

Below we shall investigate the operator D_H. Note that it acts trivially on the sections of the subbundle $\mathrm{Pol}(\bar{\tau})$ constant along the fibres (i.e., on the functions $\varphi(\alpha)$ independent of the coordinates $z \in \bar{\tau}(\alpha)$). Namely, on such functions this operator coincides with the Hamiltonian field $\mathrm{ad}(H)|_{\Lambda}$, and the formulas of Theorem 4.1 coincide with the formulas of Theorem 3.1. Now consider another cases.

(ii) *Zero Curvature and Commutative Holonomy*

Assume that there is a Hermitian subbundle $\tau^{\#} \subset \tau$ invariant with respect to parallel transports. Suppose the curvature of connection restricted on $\tau^{\#}$ is equal to zero. Then the holonomy group of the restricted connection is discrete. We assume that this group is commutative (this holds automatically if $\pi_1(\Lambda)$ is commutative). Then $\tau^{\#}$ can be expanded into a direct sum of one-dimensional invariant subbundles. Their one-dimensional fibers are common eigenspaces for commuting unitary matrices of the holonomy.

Thus we obtain the objects of type (V) from our list of basic objects over the isotropic submanifold Λ (see subsection (ix) Section 2).

So consider a one-dimensional ∇-invariant subbundle $\tau_1 \subset \tau$, on which the curvature vanishes. Fix an arbitrary initial vector $V_{10} \in \tau_1(\alpha_0)$. The holonomy on the cycle $\Gamma_j \in \pi_1(\Lambda)$ is the following

$$V_{10} \to \exp\{-2\pi i \beta_{1j}\}\; V_{10}. \tag{4.9}$$

Here

$$\beta_{1j} \overset{\mathrm{def}}{=} \frac{1}{2\pi} \oint_{\Gamma_j} \beta_1 + \frac{1}{2} w_1^{(\varepsilon_1)}(\Gamma_j), \tag{4.10}$$

the form $\beta_1 = \beta_1^{(\varepsilon_1)}$ was defined in (2.16), and the class $w_1^{(\varepsilon_1)} \in H^1(\Lambda, \mathbf{R}_2)$ is generated by the quasi-orientation ε_1 (see subsections (ix) and (vi) Section 2).

We shall say that τ_1 is *untwisted* if $w_1^{(\varepsilon_1)}(\Gamma_j) = 0$ for all cycles $\Gamma_j \in [\pi_1(\Lambda), \pi_1(\Lambda)]$ and for all cycles $\Gamma_j \in \mathrm{Tors}(H_1(\Lambda))$.

We denote by $V_1 = V_1(\alpha)$ the parallel section of τ_1, starting from V_{10}, i.e.,

$$\nabla V_1 = 0, \qquad V_1(\alpha_0) = V_{10}. \tag{4.11}$$

From formula (4.8), we see that for any integer $n_1 \in \mathbf{Z}_+$ the function

$$\varphi^{(n_1)}(\alpha, z) \stackrel{\text{def}}{=} \frac{2^{n_1/2}}{\sqrt{n_1!}} \exp\left\{-\frac{i}{2}\int_{\alpha_0}^{\alpha} \varkappa\right\} \cdot (z, \bar{V}_1(\alpha))_-^{n_1},$$

where $\alpha \in \Lambda$, $z \in \bar{\tau}(\alpha)$, satisfies the relations

$$\hat{H}\,\mathbb{I}_\Lambda(\varphi^{(n_1)}) = \mathbb{I}_\Lambda(H|_\Lambda \cdot \varphi^{(n_1)}) + O(\hbar^{3/2}),$$

if the quantization rule holds

$$\frac{1}{2\pi}\left[\hbar^{-1} p\, dq|_\Lambda - \varkappa/2 - n_1\beta_1\right] - \frac{n_1}{2} w_1^{\varepsilon_1} = 0 \qquad (\operatorname{mod} \mathbf{Z}). \tag{4.12}$$

The action of the operator $\mathbb{I}_\Lambda$ (4.5) on the function $\varphi^{(n_1)}$ is reduced to the action of the exponent $\exp\{-\breve{b}\}$ in the variables $z \in \bar{\tau}$

$$\exp\{-\breve{b}\}\ \varphi^{(n_1)} = \frac{2^{n_1/2}}{\sqrt{n_1!}} \exp\left\{-\frac{i}{2}\int_{\alpha_0}^{\alpha} \varkappa\right\}$$
$$\times \exp\left\{-\frac{1}{4} b(\alpha) \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial z}\right\} (z, \bar{V}_1(\alpha))_-^{n_1}.$$

Lemma 4.1. *There is the identity*

$$\exp\left\{-\frac{1}{4}\frac{\partial^2}{\partial \xi^2}\right\} \xi^n = \frac{1}{2^n}\mathscr{H}_n(\xi), \qquad \forall \xi \in \mathbf{C}, \qquad n \in \mathbf{Z}_+,$$

where $\mathscr{H}_n$ are Hermite polynomials

$$\mathscr{H}_n(\xi) \stackrel{\text{def}}{=} \left(2\xi - \frac{d}{d\xi}\right)^n 1 \equiv e^{\xi^2}\left(-\frac{d}{d\xi}\right)^n e^{-\xi^2}.$$

Thus we have

$$\exp\{-\breve{b}\}\ \varphi^{(n_1)} = \frac{b_1(\alpha)^{n_1}}{\sqrt{2^{n_1} n_1!}} \circ \mathscr{H}_{n_1}((z, \bar{V}_1^{\#}(\alpha))_-) \circ \exp\left\{-\frac{i}{2}\int_{\alpha_0}^{\alpha} \varkappa\right\},$$

where

$$b_1 \stackrel{\text{def}}{=} (BV_1, \bar{V}_1)_-^{1/2}, \qquad V_1^{\#} \stackrel{\text{def}}{=} V_1/\bar{b}_1,$$

where B was defined in subsection 4(i).

By substituting this expression into (4.5), we obtain the following wave function on the Riemann manifold $\mathscr{M}$:

$$\mathbb{I}_\Lambda(\varphi^{(n_1)})(q) = \frac{1}{c \cdot \sqrt{2^{n_1} n_1!}} \int_\Lambda \mathscr{H}_{n_1}((k_\alpha(y), \bar{V}_1^{\#}(\alpha))_-)|_{y = x(\alpha, q)/\sqrt{h}}$$

$$\times \exp\left\{\frac{i}{h}\Phi(\alpha, \alpha_0 \,|\, q) - \frac{i}{2}\int_{\alpha_0}^{\alpha} \varkappa\right\} \frac{b_1(\alpha)^{n_1}}{\sqrt{\Delta(\alpha)}}\, d\sigma(\alpha). \qquad (4.13)$$

COROLLARY 4.1. *Suppose the quartet* $(\Lambda, \sigma, \tau, \nabla)$ *is compatible with the symbol* H *on* $T^*\mathscr{M}$, *and* $H|_\Lambda = \lambda = const$. *Suppose a one-dimensional subbundle* $\tau_1 \subset \tau$ *is invariant and the curvature of the connection* ∇ *vanishes on* τ_1. *Also suppose that the trace of curvature is equal to zero on the whole* τ. *Then if the quantization rule* (4.12) *holds, the function* (4.13) *is a quasimode for the operator* $\hat{H} = H(q, -ih\,\partial/\partial q)$ *corresponding to the quasienergy* λ.

About the quantization rule here, the same remark as in Section 3 must be made. And precisely as in Section 3, one can exclude the quantization rule from this statement.

THEOREM 4.2. *Suppose for* $(\Lambda, \sigma, \tau, \nabla)$ *and for symbol* H *on* $T^*\mathscr{M}$ *the conditions of Corollary* 3.2-a (*or Corollary* 3.2-b) *hold. Suppose over* Λ *an additional invariant one-dimensional subbundle* $\tau_1 \subset \tau$ *is given, on which the curvature vanishes. Suppose* τ_1 *is untwisted. Then the operator* $\hat{H}$ *possesses a series of quasienergies*

$$\lambda_0 + h\left(\sum_{j=1}^{k'} m_j c^j + n_1 v_1\right), \qquad n_1 = 0, 1, 2, \ldots, \qquad m_j = 0, \pm 1, \pm 2, \ldots \quad (4.14)$$

Here

$$v_1 \stackrel{\text{def}}{=} \sum_{j=1}^{k'} c^j \beta_{1j},$$

the numbers β_{1j} *are defined in* (4.10), *the numbers* λ_0, c^j *are the same as in Section* 3. *The quasimodes corresponding to the quasienergies* (4.14) *are defined by the following integrals over the isotropic submanifold* Λ

$$\psi_m^{(n_1)}(q) = \frac{1}{c \cdot \sqrt{2^{n_1} n_1!}} \int_\Lambda \exp\left\{\frac{i}{\hbar}\,\Phi(\alpha, \alpha_0 \mid q) + i n_1 \sum_{j=1}^{k'} \beta_{1j} \int_{\alpha_0}^{\alpha} \zeta^j\right\} \varphi_m(\alpha)$$

$$\times\, \mathscr{H}_{n_1}((k_\alpha(y), \bar{V}_1^{\#}(\alpha))_-)|_{y = x(\alpha, q)/\sqrt{\hbar}} \cdot \frac{b_1(\alpha)^{n_1}}{\sqrt{\Delta(\alpha)}}\, d\sigma(\alpha). \tag{4.15}$$

Here $\mathscr{H}_{n_1}$ *is the Hermite polynomial, and the function* φ_m *is defined by formula* (3.14) *under the conditions of Corollary* 3.2-a (*or by formula* (3.17) *under the conditions of Corollary* 3.2-b).

Thus we have obtained the series (4.14) of eigenvalues for the operator $\hat{H}$ with the accuracy $O(\hbar^{3/2})$. For $n_1 = 0$, these numbers coincide with those obtained in Section 3. In particular, for $n_1 = 0$, $m = 0$, we return to the "vacuum" point of the spectrum $\lambda_0 + O(\hbar^{3/2})$ (3.18). The quantum number n_1 corresponds to excitations of the "vacuum" quasimode ψ_0. The direction of the excitation is defined by one-dimensional subbundle $\tau_1 \subset \tau$, where the curvature vanishes.

Remark 4.1. If there are several one-dimensional subbundles $\tau_1, ..., \tau_l$ such that $\nabla|_{\tau_j}$ is flat, then the same quantity of parallel sections $V_1, ..., V_l$ appears, as well as, of the functions

$$\varphi^{(n)}(\alpha, z) = \frac{2^{(n_1 + \cdots + n_l)/2}}{\sqrt{n_1! \cdots n_l!}}\, e^{-(i/2)\int_{\alpha_0}^{\alpha} \varkappa} (z, \bar{v}_1)_-^{n_1} \cdots (z, \bar{V}_l)_-^{n_l}, \tag{4.16}$$

where $n = (n_1, ..., n_l)$ is a set of integer nonnegative numbers.

Under the conditions of Corollary 4.1 *if the quantization rule holds*

$$\frac{1}{2\pi}\left[\hbar^{-1} p\, dq|_\Lambda - \varkappa/2 - \sum_{\gamma=1}^{l} n_\gamma \beta_\gamma\right] - \frac{1}{2}\sum_{\gamma=1}^{l} n_\gamma w_\gamma^{(\varepsilon_\gamma)} = 0 \qquad (\mathrm{mod}\ \mathbf{Z}) \tag{4.17}$$

the operator $\hat{H}$ *possesses the quasimode*

$$\psi^{(n)} = \mathbb{I}_\Lambda(\varphi^{(n)}), \tag{4.18}$$

corresponding to the quasienergy $\lambda = H|_\Lambda$.

Further, this statement can be transformed to an analog of Theorem 4.2. We shall obtain that the operator $\hat{H}$ has a series of quasienergy of type (4.14):

$$\lambda_0 + \hbar \sum_{j=1}^{k'} m_j c^j + \hbar(n_1 v_1 + \cdots + n_l v_l). \tag{4.19}$$

The corresponding quasimodes (4.18) will be defined by the formula of type (4.15) with a *multi-dimensional generalization of the Hermite polynomials*

$$\widetilde{\mathscr{H}}_{n_1, \ldots, n_l} \stackrel{\text{def}}{=} \exp\left\{-\frac{1}{4}\,\mathbb{B}\,\frac{\partial}{\partial\xi}\cdot\frac{\partial}{\partial\xi}\right\}\xi_1^{n_1}\cdots\xi_l^{n_l}, \tag{4.20}$$

where

$$\mathbb{B} \stackrel{\text{def}}{=} (\bar{V} \overset{\otimes}{,} \bar{V})_+^{-1*}\cdot(BV \overset{\otimes}{,} V)_+^*\cdot(\bar{V} \overset{\otimes}{,} \bar{V})_+^{-1}$$

(the asterisk denotes the transposition without complex conjugation). May be it is better to say that $\widetilde{\mathscr{H}}_{n_1, \ldots, n_l}$ is a parallel section over Λ consisting of generalized Hermitian polynomials.

Remark 4.2. Assume that the curvature of the connection vanishes on the whole bundle τ. Then τ splits into the orthogonal sum of one-dimensional invariant subbundles

$$\tau = \tau_1 \oplus \cdots \tau_r, \qquad r \equiv \dim \mathscr{M} - \dim \Lambda = {}^{\mathbf{C}}\!\dim \tau.$$

Fix a certain quasi-orientation ε_γ in each τ_γ. Then we get the quasi-orientation $\varepsilon = \varepsilon \oplus \cdots \oplus \varepsilon_\tau$ in τ. With respect to each ε_γ, we can construct a 1-form β_γ on Λ (see subsection (ix) Section 2), and with respect to ε, one can construct a 1-form β^ε (see subsection (viii) Section 2). Then

$$\beta^\varepsilon = \sum_{\gamma=1}^{r} \beta_\gamma.$$

Thus by (2.13), the quantization rule (4.17) will have the form

$$\begin{aligned}\frac{1}{2\pi}\left[\hbar^{-1}p\,dq|_\Lambda - \sum_{\gamma=1}^{r}\left(n_\gamma + \frac{1}{2}\right)\beta_\gamma\right] \\ -\frac{1}{2}\left(\sum_{\gamma=1}^{r} n_\gamma w_\gamma^{(\varepsilon_\gamma)} + \frac{1}{2}[\mu^\varepsilon]\right) = 0 \qquad (\operatorname{mod} \mathbf{Z}).\end{aligned} \tag{4.21}$$

Here by Lemma 2.3, the class $[\mu^\varepsilon] \in H^1(\Lambda, \mathbf{Z})$ possesses the property

$$[\mu^\varepsilon] = \sum_{\gamma=1}^{r} w_1^{(\varepsilon_\gamma)} + w_1(\Lambda) + w_1(\mathscr{M}) \qquad (\operatorname{mod} 2).$$

In particular, if Λ and $\mathscr{M}$ are both orientable, and if all ε_γ are trivial Euclidean subbundles, then $w_1^{(\varepsilon_\gamma)} = 0$, and the class $[\mu^\varepsilon]$ is even.

Precisely as above, the quantization rule (4.21) can be transformed into an explicit formula for quasienergies. Namely, *under the condition, that all*

τ_γ are untwisted, and the field $\mathrm{ad}(H)|_\Lambda$ *is nonresonant, a series of quasienergies of type* (4.19) (*where* $l=r$) *is obtained for the operator* $\hat{H}$. *The quasimodes are defined by the formulas of type* (4.15) (*or* (1.2)), *in which the Gaussian packets over* Λ *are framed by a multi-dimensional generalization of the Hermite polynomials* (4.20).

(iii) *Noncommutative Holonomy*

Assume that the curvature vanishes on a certain invariant Hermitian subbundle $\tau^\# \subset \tau$, but the holonomy group is non-commutative and, in particular, the fundamental group $\pi_1(\Lambda)$ is non-Abelian. We denote by G_j the holonomy matrices corresponding to the basis cycles $\Gamma_j \in \pi_1(\Lambda)$ (the matrices G_j are calculated with respect to a fixed Euclidean basis in $\tau^\#(\alpha_0)$). These matrices are unitary $G_j \in \mathbf{U}(r)$, since the connection is Hermitian. We assume that they are *orthogonal*, i.e., $G_j \in \mathbf{O}(r)$. Moreover, we assume that the *permutation relations* hold

$$G_l G_j = \sum_{s=1}^{k_1} \omega_{lj}^s G_j G_s, \tag{4.22}$$

where the coefficients ω_{lj}^s possess the following property: all the matrices

$$\Omega_j = ((\omega_{lj}^s))_{s,\, l=1,\, \ldots,\, k_1} \qquad (j=1, \ldots, k_1 \equiv \dim \pi_1(\Lambda))$$

are unitary.

Lemma 4. 2. (a) *If all the matrices* Ω_j *have a common eigenvector* $\mathbf{g} = (\mathbf{g}^1, \ldots, \mathbf{g}^{k_1}) \in \mathbf{C}^{k_1}$, *i.e.*,

$$\Omega_j \mathbf{g} = e^{-2\pi i \delta_j} \mathbf{g}, \qquad \delta_j \in \mathbf{R}, \tag{4.23}$$

then the matrix

$$M^0 = \sum_{l=1}^{k_1} \mathbf{g}^l G_l$$

permutes with all G_j *by the formula*

$$M^0 G_j = e^{-2\pi i \delta_j} G_j M^0. \tag{4.24}$$

(b) *If all the matrices* $\Omega_j \otimes \Omega_j$ *has a common eigenvector* $\tilde{\mathbf{g}} \in \mathbf{C}^{k_1} \times \mathbf{C}^{k_1}$, *i.e.*

$$(\Omega_j \otimes \Omega_j)\, \tilde{\mathbf{g}} = e^{-2\pi i \delta_j} \tilde{\mathbf{g}}, \tag{4.25}$$

then the matrix

$$M^0 = \tfrac{1}{2} \sum_{s,\, l=1}^{k_1} \tilde{\mathbf{g}}^{sl} G_s^{-1} G_l$$

permutes with all G_j *by the same formula* (4.24).

Obviously, the mapping $\Gamma_j \to e^{-2\pi i \delta_j}$ generates the character of the group $\pi_1(\Lambda)$. We assume that

a certain Nth power of this character defines a character of the group $H_1(\Lambda)$.

This means that for any two cycles $\Gamma_j, \Gamma_{j'} \in \pi_1(\Lambda)$, equivalent in the homology group, the following relation holds

$$e^{2\pi i N \delta_j} = e^{2\pi i N \delta_{j'}}.$$

Denote by $\bar{z}^{\#}$ the orthogonal projection of the vector $\bar{z} \in \tau(\alpha)$ on $\tau^{\#}(\alpha)$. Also denote by $M(\alpha)$ a solution of the problem

$$dM = M \cdot \theta^{\#} + \theta^{\#*} \cdot M, \qquad M(\alpha_0) = M^0, \tag{4.26}$$

where $\theta^{\#}$ is the form of connection $\nabla|_{\tau^{\#}}$.

Lemma 4. 3. *Under the above assumptions and under one of the conditions of Lemma* 4.2, *the function*

$$\varphi(\alpha, z) \stackrel{\text{def}}{=} (\bar{M}(\alpha)\, z^{\#} \cdot z^{\#})^N \tag{4.27}$$

is a parallel section of the bundle $\mathrm{Pol}(\bar{\tau})$ *over a universal covering of* Λ. *When running over the cycles* $\Gamma \in H_1(\Lambda)$, *this section is multiplied by* $\exp\{2\pi i \delta(\Gamma)\}$, *where* $\delta \pmod{\mathbf{Z}}$ *is a class of cohomology defined by* $\delta(\Gamma_j) =^{\text{def}} N\delta_j$.

Example 4.1. Suppose $\Lambda = \mathbf{K}^2$ is the *Klein bottle* embedded as an isotropic submanifold into $\mathbf{R}^8$. Let the subbundle $\tau = \tau^{\#}$ be two-dimensional, and the elements G_j of the holonomy group are $\mathbf{O}(2)$-matrices. The fundamental group $\pi_1(\mathbf{K}^2)$ is generated by two cycles Γ_1, Γ_2 with the relation [25]

$$\Gamma_1 = \Gamma_2 \cdot \Gamma_1 \cdot \Gamma_2.$$

Consider the $\mathbf{O}(2)$-representation of this group

$$\Gamma_1 \to G_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \Gamma_2 \to G_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.28}$$

Then $G_1 G_2 = -G_2 G_1$, i.e., in this example we have

$$\Omega_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \Omega_2 = -\Omega_1.$$

Hence there are two variants of choosing the eigenvectors $\mathbf{g}$ (4.23):

$$\mathbf{g}^I = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{g}^{II} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The corresponding matrix M^0 will be the following

$$M^{0, I} = G_1 \qquad \text{or} \qquad M^{0, II} = G_2.$$

In the first case the character $e^{2\pi i \delta_j}$ of the group $\pi_1(\mathbf{K}^2)$ has the form

$$\chi^I: \Gamma_1 \to 1, \qquad \Gamma_2 \to -1.$$

In the second case we have

$$\chi^{II}: \Gamma_1 \to -1, \qquad \Gamma_2 \to 1.$$

The mappings $(\chi^I)^2$ and χ^{II} correctly define the characters of the quotient group $H_1(\mathbf{K}^2) = \pi_1(\mathbf{K}^2)/\{\Gamma_2\}$. Thus by Lemma 4.3, we have two parallel sections of the bundle $\mathrm{Pol}(\tau)$

$$\begin{aligned} \varphi^I &\overset{\text{def}}{=} (\bar{M}^I z \cdot z)^2 \qquad \text{of power } 4; \\ \varphi^{II} &\overset{\text{def}}{=} \bar{M}^{II} z \cdot z \qquad \text{of power } 2. \end{aligned} \tag{4.29}$$

Here M^I and M^{II} are solutions of equation (4.26) with the initial conditions $M^I(\alpha_0) = G_1$ and $M^{II}(\alpha_0) = G_2$.

Further, equation (4.25) also gives two variants of solutions:

$$\tilde{\mathbf{g}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \tilde{\mathbf{g}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

They are related to

$$M^{0, III} = G_0 \equiv I \qquad \text{and} \qquad M^{0, IV} = G_3 \equiv G_1 G_2.$$

Here the character $\exp\{2\pi i \delta_j\}$ has the form

$$\chi^{III}: \Gamma_1 \to 1, \qquad \Gamma_2 \to 1$$

and

$$\chi^{IV}: \Gamma_1 \to -1, \qquad \Gamma_2 \to -1.$$

The characters χ^{III} and $(\chi^{IV})^2$ are correctly transported to the quotient group $H_1(\mathbf{K}^2)$. Thus there are two other parallel sections of the bundle $\mathrm{Pol}(\bar{\tau})$:

$$\begin{aligned}\varphi^{III} &\stackrel{\text{def}}{=} \bar{M}^{III} z \cdot z && \text{of power } 2;\\ \varphi^{IV} &\stackrel{\text{def}}{=} (\bar{M}^{IV} z \cdot z)^2 && \text{of power } 4.\end{aligned} \tag{4.29a}$$

Here M^{III} and M^{IV} are solutions of equation (4.26) with the initial conditions $M^{III}(\alpha_0)=G_0$ and $M^{IV}(\alpha_0)=G_3$.

So in the case of the Klein bottle $\Lambda=\mathbf{K}^2$, on the Hermitian bundle τ with the holonomy group (4.28) there are no invariant Hermite polynomials, but there are *four series of invariant polynomials* of the form

$$\mathscr{K}_n(\alpha, z) \stackrel{\text{def}}{=} \exp\left\{-\frac{1}{4}\, b(\alpha)\, \frac{\partial}{\partial z}\cdot\frac{\partial}{\partial z}\right\} \varphi(\alpha, z)^n, \qquad n=1, 2, ...,$$

where φ is one of the functions (4.29), (4.29a). They are related to series of quasimode of type (4.15), where $\mathscr{H}_{n_1}$ must be replaced by $\mathscr{K}_n$.

Similarly, *in the general case, on a normal bundle* τ *over an isotropic* Λ, *we get new polynomials* $\exp\{-\vec{b}\}\ \varphi^n$, *where* φ *is defined in* (4.27). *They must be substituted in integral formulas* (4.13) *instead of Hermite polynomials and give new quasimodes* (*with the same properties as in Corollary* 4.1). *The quantization rule* (4.12) *is modified as follows*

$$\frac{1}{2\pi}\,[\hbar^{-1} p\, dq\,|_{\Lambda} - \varkappa/2] + n\delta = 0 \qquad (\mathrm{mod}\ \mathbf{Z}),$$

where δ *is a class of cohomologies from Lemma* 4.2.

5. POSSIBLE MECHANISMS OF APPEARANCE OF THE QUARTET $(\Lambda, \sigma, \tau, \nabla)$

Now we consider the sufficient conditions which allow to construct effectively a compatible quartet $(\Lambda, \sigma, \tau, \nabla)$ for a given symbol H on $\mathfrak{X}=T^*\mathscr{M}$. This these conditions allow to calculate explicitly the quasimodes and quasienergies of the quantum operator $\hat{H}$ by using the formulas given in the previous sections.

(i) *Isotropic Submanifold and Invariant Measure*

Of course, the simplest version of a compact isotropic submanifold Λ is a *closed trajectory* of Hamilton flow. The investigation of the existence (and stability) of such trajectories is a classical and highly developed theory. here we shall be interested in the case $\dim \Lambda > 1$.

Suppose $\mathscr{Y}$ is a symplectic submanifold in $\mathfrak{X}$ ($\dim \mathscr{Y} = 2k$) and suppose the Hamilton field $\operatorname{ad}(H)|_{\mathscr{Y}}$ is completely integrable. Then $\mathscr{Y}$ is stratified by invariant Liouville tori. Each such torus Λ of dimension $\dim \Lambda = k$ is Lagrangian in $\mathscr{Y}$, but *isotropic* in $\mathfrak{X}$. Thus we have here a family of invariant isotropic submanifolds.

Generalizing this example, assume that a set of independent functions $f = (f_1, ..., f_k)$ is given on $\mathfrak{X}$, the intersections

$$\Lambda \stackrel{\text{def}}{=} \{f_1 = \text{const}, ..., f_k = \text{const}\} \cap \mathscr{Y}$$

is a smooth submanifold, and moreover,

$$\{f_s, f_l\}|_\Lambda = 0 \qquad \forall s, l.$$

Then Λ is also isotropic in $\mathfrak{X}$ (and Lagrangian in $\mathscr{Y}$). And the function H on $\mathfrak{X}$ has the form

$$H = \mathscr{H}(f_1, ..., f_k) + O(d_\Lambda^2), \tag{5.1}$$

where d_Λ is the distance to Λ, then the submanifold Λ is invariant with respect to the flow γ_H^t of the field $\operatorname{ad}(H)$.

In particular, we can assume that for a certain set $\mathscr{A} = (\mathscr{A}^1, ..., \mathscr{A}^n)$ of independent functions on $\mathfrak{X}$

$$\begin{aligned} \{H, \mathscr{A}^j\} &= 0, \\ \{\mathscr{A}^j, \mathscr{A}^s\} &= \Psi^{js}(\mathscr{A}), \qquad \forall j, s = 1, ..., n \end{aligned} \tag{5.2}$$

and

$$\tfrac{1}{2} \operatorname{rank} \Psi + \operatorname{corank} \Psi = \tfrac{1}{2} \dim \mathfrak{X}.$$

Then the surfaces

$$\Lambda = \{\mathscr{A}^1 = \text{const}, ..., \mathscr{A}^n = \text{const}\}$$

are isotropic, $\dim \Lambda = \operatorname{corank} \Psi$, and all Λ are invariant with respect to the flow γ_H^t. This statement is a nonlinear version of the so-called non-commutative Liouville theorem [70] (see also [29, 38, 64, 79]). In this case the remainder $O(d_\Lambda^2)$ in (5.1) vanishes; and if Λ is compact, then the functions $f_j = 1/2\pi \oint_{\Gamma_j} p\, dq$ are the action variables [72].

There is also a more general version which yields not a family, but only one isotropic submanifold. Suppose

$$\mathscr{A} = (\mathscr{A}^1, ..., \mathscr{A}^n) \qquad \text{and} \qquad \mathscr{B} = (\mathscr{B}_1, ..., \mathscr{B}_m)$$

are sets of independent functions on $\mathfrak{X}$, $n+m=\dim \mathfrak{X}$, and

$$\{\mathscr{A}^j, \mathscr{A}^s\} = \Psi^{js}(\mathscr{A}) \qquad \forall j, s$$
$$\{\mathscr{B}_j, \mathscr{A}^s\} = 0 \qquad \forall j, s.$$

We fix a point $\alpha^0 \equiv (q^0, p^0) \in \mathfrak{X}$. Then the submanifold

$$\Lambda = \{\alpha \in \mathfrak{X} \mid \mathscr{B}(\alpha) = \mathscr{B}(\alpha^0), \mathscr{A}(\alpha) = \mathscr{A}(\alpha^0)\}$$

is `isotropic` and $\dim \Lambda = \operatorname{corank} \Psi(\zeta^0)$. Moreover, if

$$\begin{aligned} \{H, \mathscr{A}^j\} &= 0 \qquad \forall j \quad \text{everywhere on } \mathfrak{X}, \\ \{H, \mathscr{B}_l\}(\alpha^0) &= 0 \qquad \forall l, \end{aligned} \tag{5.3}$$

then Λ is invariant with respect to the flow γ_H^t.

In the cases (5.1) and (5.3) we say that the field $\operatorname{ad}(H)$ has *infinitesimal symmetries* [47–49]. The restriction of this field on Λ can be written in the form

$$\operatorname{ad}(H)|_\Lambda = \sum_{j=1}^{k} \omega^j \mathscr{D}_j, \tag{5.4}$$

where $\omega^j = \text{const}$, and $\{\mathscr{D}_j\}$ is a certain basis of vector fields on Λ:

$$[\mathscr{D}_j \mathscr{D}_s] = \sum_{l=1}^{k} \lambda_{js}^l \mathscr{D}_l, \tag{5.5}$$

λ_{js}^l are real smooth functions on Λ.

In the case (5.3) $\lambda_{js}^l = \text{const}$ are the structure constants of a certain *Lie algebra* $\mathfrak{g}$, and $\mathscr{D}_j$ are generators of action of the corresponding *Lie group* $\mathfrak{G}$. At the points in general position, where $\operatorname{rank} \Psi(\mathscr{A}(\alpha^0))$ is maximal, this algebra is abelian: $\lambda_{js}^l = 0$. In particular, in the version (5.2) the algebra $\mathfrak{g}$ is abelian beforehand.

We denote by $\{\eta^s\}$ the basis of 1-forms on Λ dual to the basis $\{\mathscr{D}_l\}$, $\langle \eta^s, \mathscr{D}_l \rangle = \delta_l^s$.

LEMMA 5.1. *Suppose $\lambda_{js}^l = \text{const}$. Then*

(a) *The form $\operatorname{tr}(\lambda \cdot \eta) = \sum_{l,\, j=1}^{k} \lambda_{jl}^l \eta^j$ is exact on Λ, i.e., the global function*

$$\Delta_\Lambda(\alpha) = \exp\left\{\int_{\alpha_0}^{\alpha} \operatorname{tr}(\lambda \cdot \eta)\right\}; \tag{5.6}$$

is defined.

(b) *If the metric with respect to which all the fields* $\mathscr{D}_i$ *are geodesic is fixed on* Λ, *and if to this metric the subbundle* π *is assigned according to Lemma* 2.1, *then the function* (5.6) *coincides with* (2.7).

(c) *The measure on* Λ

$$d\sigma \stackrel{\text{def}}{=} \Delta_\Lambda\, |\eta^1 \wedge \cdots \eta^k| \tag{5.7}$$

is invariant with respect to the flow γ_H^t *of the field* (5.4).

Note that in situation of Lemma 5.1 $\Lambda \approx \mathfrak{G}/\mathfrak{G}_d$, where $\mathfrak{G}_d$ is a certain discrete subgroup in the Lie group $\mathfrak{G}$. The fields $\mathscr{D}_l$ and the forms η^s are analogs of left-invariant fields and forms on the Lie group. The function (5.6) is an analog of the group modular function. The measure (5.7) is an analog of the right Haar measure.

(ii) *Normal Positive Subbundle and Adapted Connection*

Suppose an isotropic γ_H^t-invariant submanifold $\Lambda \subset \mathfrak{X}$ is given. Then we need a positive subbundle $\tau \subset {}^{\mathbf{C}}\mathring{E}$ and a Hermitian connection on τ adapted to the first variation equation ${}^{\tau}\mathrm{Var}(H)$. How can they be constructed?

We shall show that this question is reduced to the problem on existence of a connection on $\mathring{E}$ with appropriate properties of adaptation and stability.

Assume that on $\mathring{E}$ there is a certain *symplectic* connection ∇^{sym} with zero curvature adapted to the first variation equation of the field $\mathrm{ad}(H)$. This means that

(i) the parallel transport

$$G(\alpha, \alpha_0)\colon \mathring{E}(\alpha_0) \to \mathring{E}(\alpha) \tag{5.8}$$

with respect to ∇^{sym} along any curve $\{\alpha_0 \to \alpha\} \subset \Lambda$ is a symplectic transformation of fibers;

(ii) $[\nabla_u^{\text{sym}}, \nabla_v^{\text{sym}}] = \nabla_{[u,v]}^{\text{sym}}$ for any vector fields u, v on Λ;

(iii) the parallel transport coincides with the differential of the flow γ_H^t, i.e.,

$${}^{\mathring{E}}\widetilde{\nabla}^{\text{sym}}_{\mathrm{ad}(H)} = {}^{\mathring{E}}\mathrm{ad}^{\#}(H),$$

where to the right we have the projection along $T\Lambda$ onto $\mathring{E}$ of the field $\mathrm{ad}^{\#}(H)$, defined on $T_\Lambda\mathfrak{X}$, and to the left we have a similar projection of the covariant derivative $\widetilde{\nabla}^{\text{sym}}$ onto the bundle $\mathrm{Pol}(\mathring{E})$ (for details, see Appendix B below and the paper [51]).

We denote by θ the form of the connection ∇^{sym} in a certain local basis of sections $\{\mathring{X}_\nu\} \subset \Gamma^\infty(\mathring{E})$. By definition, a parallel transport on $\mathring{E}$ is the map $G = G(\alpha, \alpha_0)$ (5.8), satisfying the equation

$$\nabla^{\mathrm{sym}} G \equiv dG + \theta(\alpha)\, G = 0, \qquad G|_{\alpha=\alpha_0} = I. \tag{5.9}$$

Hence, the symplecticity condition (i), the zero curvature condition (ii), and the adaptation condition (iii) can be written as a triple of equations on Λ

$$\begin{aligned} \theta^* \mathscr{J} + \mathscr{J}\theta - d\mathscr{J} &= 0, \\ d\theta + \theta \wedge \theta &= 0, \\ \langle \theta, \mathrm{ad}(H)|_\Lambda \rangle + \mathrm{Var}(H) &= 0. \end{aligned} \tag{5.10}$$

Here $\mathscr{J} = \omega(X \overset{\otimes}{,} X)^*$ is the tensor of symplectic structure on $\mathring{E}$ written in the basis $\{\mathring{X}_\nu\}$, the asterisk denotes the transposition. The matrix $\mathrm{Var}(H)$ of the first variation equation is defined as follows

$$[X_\nu, \mathrm{ad}(H)] = \sum_{\mu=1}^{2r} \mathrm{Var}(H)^\mu_\nu\, X_\mu + (\text{section of } T\Lambda),$$

or in the explicit from

$$\mathrm{Var}(H)^\mu_\sigma = \sum_{\nu=1}^{2r} \mathscr{J}^{-1\nu\mu}(D^2 H X_\nu \cdot X_\sigma - \omega(X_\nu, \mathrm{ad}(H)\, X_\sigma)). \tag{5.11}$$

Remark 5.1. In the situation (5.4), (5.5) we can represent the connection form θ as $\theta = \sum_{i=1}^k \theta_i \eta^i$, and then reduce (5.10) to the following system of equations for the coefficients θ_i

$$\begin{aligned} \theta_i^* \mathscr{J} + \mathscr{J}\theta_i - \mathscr{D}_i(\mathscr{J}) &= 0, \\ \mathscr{D}_i \theta_j - \mathscr{D}_j \theta_i - \sum_{s=1}^k \lambda^s_{ij} \theta_s + [\theta_i, \theta_j] &= 0, \\ \sum_{i=1}^k \omega^i \theta_i + \mathrm{Var}(H) &= 0. \end{aligned} \tag{5.12}$$

We denote by $\mathrm{Ad}(\alpha)$ the solution of the matrix Cauchy problem over the universal covering of Λ:

$$d\,\mathrm{Ad}^s_m = \sum_{l=1}^k (\eta \cdot \lambda)^l_m\, \mathrm{Ad}^s_l, \qquad \mathrm{Ad}(\alpha_0) = I.$$

Fix some real constants $h^1, ..., h^k$ and define

$$W \equiv \mathrm{Var}(H), \qquad L = -\sum_{s,l=1}^{k} h^s (\mathrm{Ad}^{-1})^l_s \, \theta_l,$$

$$\frac{\partial}{\partial t} = \sum_{l=1}^{k} \omega^l \mathscr{D}_l, \qquad \frac{\partial}{\partial \tau} = \sum_{s,l=1}^{k} h^s (\mathrm{Ad}^{-1})^l_s \, \mathscr{D}_l.$$

Then from the last two equations of the system (5.12) we obtain the following *equation of the Lax type*

$$\frac{\partial W}{\partial \tau} - \frac{\partial L}{\partial t} + [W, L] = 0.$$

But it is necessary to stress that this equation is not equivalent to (5.12). In the system (5.12) the dimension and the topology of an isotropic submanifold Λ play an essential role (What distinguishes our case from the well-known situation [24]).

Now we assume that the system (5.10) or (5.12) is solved (see below Section 6), i.e., an adapted symplectic connection with zero curvature is given on the bundle $\mathring{E}$ over Λ. Then the holonomy operators in the fiber over a point α_0 are symplectic operators, i.e., they belong to $\mathrm{Sp}(E_{\alpha_0}, \mathbf{R})$.

An operator from $\mathrm{Sp}(E_{\alpha_0}, \mathbf{R})$ is called *stable* if it can be diagonalized and its spectrum lies on the unit circle in $\mathbf{C}$; see, for example, [5]. A holonomy group will be called *stable* if it is commutative and all its generators are stable.

The first variation equation of a field $\mathrm{ad}(H)$ will be called *stable over an isotropic submanifold* Λ, if this equation is adaptable on $\mathring{E}$ to a symplectic connection with zero curvature and stable holonomy.

LEMMA 5.2. (a) *A set of commutative stable symplectic operators in a real-valued symplectic space has a common eigenbasis* (*in the complexification of this space*); *a half of the vectors in this basis are positive, and a half are negative with respect to the Kählerian structure* $(..., ...)_+$.

(b) *If the holonomy of the connection* ∇^{sym} *on* $\mathring{E}$ *is stable, then there is a decomposition into the direct sum*

$$\mathring{E} = \mathring{E}_1 \oplus \cdots \oplus \mathring{E}_r, \qquad \dim \mathring{E}_\gamma = 2, \qquad \mathring{E}_\gamma \text{ is symplectic } (\gamma_1, ..., r).$$

Here each subbundle $\mathring{E}_\gamma$ *is* ∇^{sym}*-invariant and the holonomy is stable on it. Moreover,*

$${}^{\mathbf{C}}\mathring{E}_\gamma = \tau_\gamma \oplus \bar{\tau}_\gamma, \qquad {}^{\mathbf{C}}\dim \tau_\gamma = 1, \qquad \tau_\gamma \text{ is positive},$$

and each τ_γ *is* ∇^{sym}*-invariant. The subbundle*

$$\tau \stackrel{\mathrm{def}}{=} \tau_1 \oplus \cdots \tau_r$$

is the normal positive subbundle over Λ, *and the restriction* $\nabla \stackrel{\mathrm{def}}{=} \nabla^{\mathrm{sym}}|_\tau$ *is a Hermitian connection on* τ *with zero curvature.*

(c) *If the first variation equation of the field* $\mathrm{ad}(H)$ *is stable over* Λ *with respect to the connection* ∇^{sym}, *then all one-dimensional positive subbundles* τ_γ *from statement* (a) *are invariant with respect to the first variation equation.*

(d) *If in addition to the conditions of* (b) *the Chern class* $c_1(\tau_\gamma)$ *is zero in* $H^2(\Lambda, \mathbf{Z})$, *then* $\mathring{E}_\gamma$ *is trivial. In this case the first variation equation has a solution of the Floquet type*:

$$X_\gamma(t) = e^{-i\int_{\alpha_0}^{\alpha(t)} \beta_\gamma} Y_\gamma(\alpha(t)). \tag{5.13}$$

Here Y_γ *is a global section of* τ_γ *with unit norm,* $\alpha(t)$ *is a trajectory of the field* $\mathrm{ad}(H)$ *on* Λ, *and* $i\beta$ *is the connection form of* $\nabla|_{\tau_\gamma}$, *i.e.*,

$$\nabla Y_\gamma = i\beta_\gamma Y_\gamma,$$

or

$$i\beta_\gamma = (dY_\gamma, Y_\gamma)_+ + (\theta Y_\gamma, Y_\gamma)_+. \tag{5.14}$$

(*see subsection* (ix) *Section* 2).

Thus we have reduced the problem about the existence of the quartet $(\Lambda, \sigma, \tau, \nabla)$, compatible with a given symbol H, to two problems.

Problem 1°: the existence of infinitesimal symmetries for H of type (5.1).

Problem 2°: solving the system (5.12)

$$\text{symplecticity} + \text{zero curvature} + \text{adaptation}.$$

Now we consider certain particular cases, where Problem 2° can be either resolved explicitly, or reduced to Problem 1°.

6. CERTAIN SOLUTIONS OF EQUATIONS OF ZERO CURVATURE OVER ISOTROPIC SUBMANIFOLD

(i) *Canonical Actions of Pseudogroups*

Suppose there is a Hamilton action of a Lie group $\mathfrak{G}$ on a symplectic manifold $\mathfrak{X}$, and an isotropic submanifold Λ is one of $\mathfrak{G}$-orbits. Suppose $u_1, \ldots, u_k$ are generators of this action in a neighborhood of Λ. Then

$$
\begin{gathered}
u_j|_\Lambda \in \Gamma^\infty(T\Lambda), \qquad L_{u_j}\omega = 0, \\
[u_j, u_s] = \sum_{l=1}^{k} \lambda_{js}^l u_l, \qquad \text{where} \quad \lambda_{js}^l = \text{const.}
\end{gathered} \tag{6.1}
$$

The connection ∇^{sym} on $\mathring{E}$ is defined by the canonical Bott procedure [12]

$$
\nabla_u^{\text{sym}}(\mathring{X}_\nu) \stackrel{\text{def}}{=} [u, \mathring{X}_\nu]. \tag{6.2}
$$

Here $\{X_\nu\}$ is a local basis in E, and u is any vector field tangent to $\mathfrak{G}$-orbits (i.e., a linear combination of generators u_j).

Note that the right-hand side of (6.2) taken with the sign "minus" coincides with the first variation of the field u (see Appendix B), namely,

$$
[X_\nu, u] = \sum_{\mu=1}^{2r} \text{Var}_\nu^\mu \cdot X_\mu + (\text{section of } T\Lambda). \tag{6.2a}
$$

Thus the matrix θ of the form of connection ∇^{sym} and the matrix of the first variation differ from each other only by the sign

$$
\langle \theta_\nu^\mu, u \rangle = -\text{Var}_\nu^\mu
$$

This means that the connection ∇^{sym} is adapted to any vector field u tangent to $\mathfrak{G}$-orbits.

From (6.1), (6.2), one can easily see that the curvature of the connection ∇^{sym} is equal to zero.

In order to verify whether ∇^{sym} is symplectic, we represent the fields X_ν in the form

$$
X_\nu = \text{ad}(\Phi_\nu), \tag{6.3}
$$

where $\{\Phi_\nu\}$ is a certain set of independent functions in a neighborhood of a given point on Λ, and

$$
\Phi_\nu|_\Lambda = 0, \qquad \det((\{\Phi_\nu, \Phi_\mu\}))|_\Lambda \neq 0. \tag{6.3a}
$$

Then the symplecticity of the connection ∇^{sym} is equivalent to the following identities

$$
u(\{\Phi_\nu, \Phi_\mu\})|_\Lambda = \{u(\Phi_\nu), \Phi_\mu\}|_\Lambda + \{\Phi_\nu, u(\Phi_\mu)\}|_\Lambda, \qquad \forall \nu, \mu \quad \forall u
$$

or

$$
(d\pi_u)(\text{ad}(\Phi_\nu), \text{ad}(\Phi_\mu))|_\Lambda = 0. \tag{6.4}
$$

Here the form π_u is dual to the field u with respect to the symplectic structure, i.e.,

$$u(f)=\langle \pi_u, \mathrm{ad}(f)\rangle, \qquad \forall f\in C^\infty(\mathfrak{X}).$$

Since the generators u_j are Hamiltonian, then $d\pi_{u_j}=0$, i.e., (6.4) holds for $u=u_j$. And it is easy to see that (6.4) holds also for any linear combination of generators.

So we have proved

LEMMA 6.1. *Suppose the orbit Λ of the Hamiltonian action of the Lie groups is isotropic. Then on normal symplectic bundle $\mathring{E}$ over Λ the Bott connection has zero curvature, is symplectic, and adapted to any vector field whose flow leaves Λ invariant.*

Of course, this statement, may be except the last property of adaptation, is well-known (see [19, 61, 87]). But the approach to its proof given above can be transformed to a more general case.

THEOREM 6.1. *The statement of Lemma 6.1 holds if instead of a Hamiltonian action of a Lie group a canonical action of a pseudogroup is given in the sense of* [39].

Actually, in this case the generators of the action will be as follows

$$u_j=-\sum_{l=1}^{n}\pi_{jl}(\mathscr{A})\,\mathrm{ad}(\mathscr{A}^l). \tag{6.5}$$

Here $\mathscr{A}\colon\mathfrak{X}\to N$ is a Poisson momentum mapping (its base N is a certain Poisson manifold), and the forms

$$\pi_j(\xi)\overset{\mathrm{def}}{=}\sum_{l=1}^{n}\pi_{jl}(\xi)\,d\xi^l, \qquad \xi\in N$$

define a basis in the space $\mathscr{F}^1(N)$ of 1-forms on N

$$d\pi_j=-\tfrac{1}{2}\sum_{l,m}\mu_j^{lm}\pi_l\wedge\pi_m, \qquad \mu_j^{lm}\in C^\infty(N). \tag{6.6}$$

The commutators of generators (6.5) are as follows

$$[u_j,u_l]=\sum_m\lambda_{jl}^m(\mathscr{A})\,u_m, \qquad \lambda_{jl}^m\in C^\infty(N). \tag{6.7}$$

We note that λ_{jl}^m are structure functions of a pseudoalgebra, i.e.,

$$[\pi_j, \pi_l]_{\mathscr{F}^1(N)} = \sum_m \lambda_{jl}^m \pi_m$$

(details see in [39]).

By Definition (6.2) and by (6.7), we obtain that the curvature is equal to zero:

$$([\nabla_{u_j}^{\mathrm{sym}}, \nabla_{u_l}^{\mathrm{sym}}] - \nabla_{[u_j, u_l]}^{\mathrm{sym}})\, X_\nu = [[u_j, u_l], X_\nu] - \sum_m \lambda_{jl}^m [u_m, X_\nu]$$

$$+ (\text{section of } T\Lambda) = 0 + (\text{section of } T\Lambda).$$

Further, by (6.5), we have $\pi_{u_j} = \mathscr{A}^* \pi_j$, and (6.6) gives

$$(d\pi_{u_j})(\mathrm{ad}(\Phi_\mu), \mathrm{ad}(\Phi_\nu)) = \sum_{l,m} \mu_j^{lm}(\mathscr{A}) \langle \pi_{u_m}, \mathrm{ad}(\Phi_\mu)\rangle \cdot \langle \pi_{u_l}, \mathrm{ad}(\Phi_\nu)\rangle$$

$$= \sum_{l,m} \mu_j^{lm}(\mathscr{A})\, u_m(\Phi_\mu)\, u_l(\Phi_\nu).$$

This yields (6.4) since $u_m(\Phi_\mu)|_\Lambda = 0$, $\forall m, \mu$. Thus the connection is symplectic. This statement completes the proof.

Remark 6.1. In contrast to the Hamilton version, the forms π_j in (6.6) are nonclosed if $\mu_j^{lm} \neq 0$. In particular, if $\mu_j^{lm} = \text{const}$, then N is supplied with a Lie group structure, for which the forms ε_j are left-invariant. If in addition $\lambda_{jl}^m = \text{const}$, it turns out that the Poisson bracket on N is multiplicative (i.e., $N \times N \to N$ is a Poisson mapping), or differs from the multiplicative one by a "cocycle". Then on the group $\mathfrak{G}$ with structural constants λ_{jl}^m a multiplicative Poisson bracket also appears, and the action of $\mathfrak{G}$ on $\mathfrak{X}$ (i.e., the mapping $\mathfrak{G} \times \mathfrak{X} \to \mathfrak{X}$), as well as, the "coadjoint" action of $\mathfrak{G}$ on N (i.e., the mapping $\mathfrak{G} \times N \to N$) are Poisson [21, 39, 63, 77].

We recall that a group with multiplicative bracket is called a Poisson group; this is the classical limit of the quantum group. Thus we have

COROLLARY 6.1. *Suppose there is a Poisson action of a Poisson group $\mathfrak{G}$ on a symplectic manifold $\mathfrak{X}$. Suppose one of the $\mathfrak{G}$-orbits $\Lambda \subset \mathfrak{X}$ is an isotropic submanifold (for example, this holds for an orbit that pulls into a unit of the group N under momentum mapping $\mathscr{A}: \mathfrak{X} \to N$). Then the Bott connection* (6.2) *on the normal symplectic bundle $\mathring{E}$ over Λ possesses all the properties mentioned in Lemma* 6.1.

(ii) *Connection Generated by Coisotropic Germ*

The construction of the previous subsection was essentially based on the existence of a coisotropic submanifold $\mathscr{K}$, in which Λ is one of isotropic fibers (in notation of Theorem 6.1 $\mathscr{K} \stackrel{\text{def}}{=} \mathscr{A}^{-1}(\xi_0)$, where $\xi_0 = \mathscr{A}(\Lambda)$; $\mathscr{K}$ is fibered by isotropic orbits of the pseudogroup action). But in fact, it is sufficient to consider only infinitesimal coisotropic germs over Λ. In this case we also obtain the symplectic connection over Λ, but may be with nonzero curvature. The condition of zero curvature will be an additional requirement imposed on the coisotropic germ.

The submanifold $\mathscr{K} \subset \mathfrak{X}$ will be called *coisotropic over* Λ if

$$\begin{aligned} &\text{(a)}\quad \Lambda \subset \mathscr{K}, \\ &\text{(b)}\quad \operatorname{Ker} \omega|_{T_\alpha \mathscr{K}} = T_\alpha \Lambda, \qquad \forall \alpha \in \Lambda, \\ &\text{(c)}\quad \operatorname{codim} \mathscr{K} = \dim \Lambda. \end{aligned}$$

If in addition it is known that $\mathscr{K}$ is a coisotropic submanifold in $\mathfrak{X}$, we shall call it a *coisotropic extension of* Λ.

Let a submanifold $\mathscr{K}$ coisotropic on Λ be given. Then it is locally defined by following equations

$$\mathscr{K} = \{f_1 = \cdots = f_k = 0\}, \qquad k = \dim \Lambda,$$

where $f = \{f_j\}$ is a set of independent functions in a neighborhood of a given point on Λ. Consider also the set of functions $\Phi = \{\Phi_\nu\}$ in this neighborhood, as well as, the basis $X_\nu = \operatorname{ad}(\Phi_\nu)$ in E; see (6.3), (6.3a). Then

$$\{f_i, f_j\} = \sum_{s=1}^{k} \lambda_{ij}^s f_s + O(f^2 + \Phi^2), \tag{6.8}$$

where λ_{ij}^s are smooth functions.

Introduce the fields $\mathscr{D}_j \stackrel{\text{def}}{=} \operatorname{ad}(f_j)|_\Lambda$ and 1-forms dual to them $\{\eta^i\} \in \mathscr{F}^1(\Lambda)$ (see subsection (i) Section 5). Also define the matrices

$$\mathscr{J} = ((\mathscr{J}_{\mu\nu})), \qquad \mathscr{J}_{\mu\nu} \stackrel{\text{def}}{=} \{\Phi_\nu, \Phi_\mu\}|_\Lambda,$$

and

$$\begin{aligned} \theta_j &= ((\theta^\mu_{j\nu})), &\qquad \theta_j &\stackrel{\text{def}}{=} \mathscr{J}^{-1}\{\Phi \overset{\otimes}{,} \{\Phi, f_j\}\}|_\Lambda, \\ \Omega_{ij} &= ((\Omega^{\;\mu}_{ij\,\nu})), &\qquad \Omega_{ij} &\stackrel{\text{def}}{=} -\sum_{s=1}^{k} \lambda_{ij}^s \theta_s + \mathscr{J}^{-1}\{\Phi \overset{\otimes}{,} \{\Phi, \{f_i, f_j\}\}\}|_\Lambda \end{aligned}$$

Here we used the following tensor notation $(\Phi \otimes \Psi)_{\nu\mu} = \Phi_\nu \Psi_\mu$.

Now consider the 1-form

$$\theta \stackrel{\text{def}}{=} \sum_{i=1}^{k} \theta_i \eta^i \tag{6.9}$$

and the 2-form

$$\Omega \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i,\, j=1}^{k} \Omega_{ij} \eta^i \wedge \eta^j. \tag{6.10}$$

THEOREM 6.2. *Each submanifold $\mathscr{K}$ coisotropic on Λ generates a linear connection on the norm symplectic bundle $\mathring{E}$ over Λ. In the local basis $\{\mathring{X}_\nu\}$ the form of this connection is defined by formula* (6.9). *The corresponding form of curvature $\Omega = d\theta + \theta \wedge \theta$ is calculated by formula* (6.10). *This is a unique symplectic connection $\mathring{E}$ adapted to all Hamiltonian fields* $\mathrm{ad}(F)$, *where $F|_{\mathscr{K}} = const$. Its curvature vanishes $\Omega = 0$ iff for any two functions F', F'' constant on $\mathscr{K}$ the following condition holds*

$$\{F', F''\}\,|_{\mathscr{K}} = O(d_\Lambda^3). \tag{6.11}$$

Here d_Λ denotes the distance from a point on $\mathscr{K}$ to Λ.

A variant of this theorem was obtained in [83, 84]. About the given version and its applications see [51]. The proof follows directly from the relations

$$\{f_i, \Phi_\nu\} = \sum_{\mu=1}^{2r} \theta^{\mu}_{i\nu} \Phi_\mu + O(\Phi^2 + f),$$

$$\{f_i, f_j\} = \frac{1}{2} \sum_{\nu,\, \mu=1}^{2r} R^{\nu\mu}_{ij} \Phi_\nu \Phi_\mu + O(\Phi^3 + f),$$

where $R_{ij} = -\Omega_{ij} \mathscr{J}^{-1}$.

Remark 6.2. We see that it is sufficient to consider only *germs* of submanifolds coisotropic on Λ. Such two germs $\mathscr{K}_1$ and $\mathscr{K}_2$ can be called *equivalent* $\mathscr{K}_1 \sim \mathscr{K}_2$ if the connections on $\mathring{E}$ corresponding to them coincide. The quotient class can be called a *coisotropic germ* over Λ. A germ is called *flat* if the curvature of the corresponding connection is equal to zero. In particular, if a germ contains a coisotropic submanifold, it is flat beforehand.

We conclude:

Flat coisotropic germs over Λ yield the solutions of the system: *symplecticity + zero curvature + adaptation.*

Coisotropic germs and more subtle objects: "adapted symplectic germs", as well as the obstructions to the existence of Poincaré map in the case $\dim \Lambda > 1$ are considered in detail in [52a].

Remark 6.3. The connection ∇^{sym} defined in Theorem 6.2 satisfies the relation (6.2) for the basic fields $\mathrm{ad}(f_j)$, but not for all fields u tangent to Λ. If $\mathscr{K}$ is coisotropic not only on Λ, but globally, then it possesses an isotropic foliation and Λ is one of the leaves. In this case the relation (6.2) holds for all fields u tangent to isotropic leaves, and the connection ∇^{sym} is the same as in the construction by Weinstein [87], Dazord [19], and Lichnerowicz [61].

(iii) *Equations of Zero Curvature in Adiabatic Approximation*

Over an abstract manifold Λ we consider the following equations for the matrix-valued 1-form θ over Λ

- of *zero curvature*

$$d\theta + \theta \wedge \theta = 0, \tag{6.12a}$$

- of *symplecticity*

$$\theta^* \mathscr{J} + \mathscr{J}\theta - d\mathscr{J} = 0, \tag{6.12b}$$

- of *adaptation*

$$\langle \theta, u \rangle + \frac{1}{\epsilon} W = 0. \tag{6.12c}$$

Here $\epsilon \in [0, 1]$ is a small parameter. In (6.12b) a smooth family $\mathscr{J} = \{\mathscr{J}(\alpha) \mid \alpha \in \Lambda\}$ of nondegenerate antisymmetric matrices is fixed

$$\mathscr{J}^* = -\mathscr{J}, \qquad \det \mathscr{J} \neq 0. \tag{6.13}$$

In (6.12c) u is a given vector field over Λ, and W is a given family of matrices

$$W = S(\alpha, \epsilon) - \frac{\epsilon}{2} \mathscr{J}(\alpha)^{-1} \cdot u(\mathscr{J}(\alpha)), \qquad \alpha \in \Lambda, \tag{6.14}$$

where S is a smooth (in α and in ϵ) family of Hamiltonian matrices

$$S^* \mathscr{J} + \mathscr{J} S = 0. \tag{6.15}$$

Remark 6.4. So we now assume that a bundle over Λ, where the form θ determines the connection, is trivial; i.e., the basis $\{X_\nu\}$ in (5.11) is global. The matrix W plays the role of the variation matrix $\mathrm{Var}(H)$ in this basis. Moreover, in this case we have $\mathrm{ad}(H)|_\Lambda = \epsilon \cdot u$, i.e., the frequencies of the field $\mathrm{ad}(H)|_\Lambda$ are supposed to be small $\sim \epsilon$. This is the *adiabaticity condition.* It means that the motion along Λ is slower than the normal oscillations generated by an "external field" S. Or on the contrary, by dividing the expressions in (6.14) by ϵ, one can get the case of a large "external field" $(1/\epsilon)\, S$. By the way, in notation (5.11) we have an explicit formula

$$S^\mu_\sigma = \sum_{\nu=1}^{2r} \mathscr{I}^{-1\nu\mu} \left[D^2 H X_\nu \cdot X_\sigma - \frac{\epsilon}{2} (\omega(X_\nu, u(X_\sigma)) + \omega(X_\sigma, u(X_\nu))) \right].$$

Now the only important fact for us is the presence of a parameter ϵ in system (6.12). This allows to construct an approximate solution with accuracy to mod $O(\epsilon^\infty)$. More precisely, system (6.12) can be reduced (mod $O(\epsilon^\infty)$) to scalar equations on Λ. Sometimes these equations can be easily solved (for example, in the case of a nonresonant field u). Then we get effective formulas for symplectic connection over Λ with zero curvature. In more complicated cases the equations obtained yield an interesting problem for further investigations.

First of all, we recall that the spectrum of a Hamiltonian matrix S (in the sense of (6.15)) may include only the following points:

- pairs of pure imaginary conjugate numbers;
- the number 0;
- pairs of pure real numbers with opposite signs;
- tedrads of numbers λ, $-\lambda$, $\bar{\lambda}$, $-\bar{\lambda}$ with nonzero real and imaginary parts.

The last two variants mean that the unstability is possible. We shall consider the first variant only.

Assume that the following condition of *strong stability* holds:

the matrices $S = S(\alpha, \epsilon)$ *have only imaginary eigenvalues* ik_γ *of multiplicity* 1, *smoothly depending on* $\alpha \in \Lambda$ *and on* $\epsilon \in [0, 1]$.

Let us numerate by $\gamma = 1, ..., r$ the eigenspaces of S on which the Kählerian product

$$((v, v))_+ = \frac{i}{2} \mathscr{I} v \cdot \bar{v}$$

is positively defined; and numerate by $\gamma = -1, ..., -r$ the conjugate eigenspaces on which the product $((..., ...))_-$ is positively defined. Denote by y_γ the eigenvectors

$$Sy_\gamma = ik_\gamma y_\gamma, \qquad \gamma = \pm 1, ..., \pm r,$$

and normalize them as follows

$$\begin{aligned} ((y_\gamma, y_\gamma))_\pm &= 1, \qquad & \pm &\equiv \operatorname{sgn}(\gamma), \\ k_{-\gamma} &= -k_\gamma, \qquad & y_{-\gamma} &= \bar{y}_\gamma. \end{aligned} \tag{6.16}$$

LEMMA 6.3. *Suppose the field u on Λ is nonresonant. Then for $\gamma = \pm 1, ..., \pm r$ there exist real closed 1-forms a_γ, a'_γ over Λ such that*

$$\begin{aligned} \langle a_\gamma, u\rangle + k_\gamma &= 0, \qquad & a_{-\gamma} &= -a_\gamma, \\ \langle a'_\gamma, u\rangle - \operatorname{Im}((u(y_\gamma), y_\gamma))_{\operatorname{sgn}(\gamma)} &= 0, \qquad & a'_{-\gamma} &= -a'_\gamma. \end{aligned} \tag{6.17}$$

To prove this statement we note that any equation

$$\langle a, u\rangle = \mathcal{N}, \qquad \mathcal{N} \in C^\infty(\Lambda)$$

has a solution $a \in \mathcal{F}^1(\Lambda)$ such that

$$da = 0, \qquad a = \sum_{j=1}^{k'} \zeta^j e_j + \mathcal{S}.$$

Here $\{\zeta^j\}$ is the basis of closed 1-forms on Λ (3.10), the numbers e_j satisfy the relation $\sum_{j=1}^{k'} \text{const}^j \cdot e_j = \langle \mathcal{N} \rangle$, where $\langle \mathcal{N} \rangle \overset{\text{def}}{=} \int_\Lambda \mathcal{N}(\alpha)\, d\sigma(\alpha)$, and const^j are the frequencies of the field u in the sense of Section 3. The function $\mathcal{S}$ is a solution of an equation on Λ similar to (3.11)

$$u(\mathcal{S}) = \mathcal{N} - \langle \mathcal{N} \rangle.$$

This equation is solvable since the field u is nonresonant.

The second column of conditions (6.17) can be satisfied by (6.16). This completes the proof.

Now introduce the matrix $\mathbb{P}_\gamma$ of the projection on the eigenspace with the number γ along all the other eigenspaces of S. We have by definition

$$\mathbb{P}_\gamma v = y_\gamma \cdot ((v, y_\gamma))_{\operatorname{sgn}(\gamma)}.$$

Let denote

$$\mathbb{P}'_\gamma = \frac{1}{2} \sum_{\mu \neq \gamma} \frac{1}{k_\mu - k_\gamma} (\mathbb{P}_\mu \mathcal{J}^{-1} u(\mathcal{J})\, \mathbb{P}_\gamma + \mathbb{P}_\gamma \mathcal{J}^{-1} u(\mathcal{J})\, \mathbb{P}_\mu),$$

$$\mathbb{A}_\gamma = \sum_{\mu \neq \gamma} \frac{1}{k_\mu - k_\gamma} [\mathbb{P}_\mu, u(\mathbb{P}_\gamma)],$$

$$\tilde{a}_\gamma = a'_\gamma + i \cdot ((dy_\gamma, y_\gamma))_{\mathrm{sgn}(\gamma)}.$$

And then define the following matrix-valued 1-form on Λ, similar to the well-known Berry form

$$\theta \stackrel{\mathrm{def}}{=} \sum_{\gamma=-r}^{r} \left(i \left(\frac{a_\gamma}{\epsilon} + \tilde{a}_\gamma \right) \mathbb{P}_\gamma + \mathbb{P}_\gamma\, d\mathbb{P}_\gamma + i a_\gamma (\mathbb{P}'_\gamma + \mathbb{A}_\gamma) \right) + O(\epsilon). \quad (6.18)$$

THEOREM 6.3. *Suppose on Λ the matrix-valued function S is strongly stable, and the field u is nonresonant. Then formula* (6.18) *gives a real solution of the system*: *zero curvature + symplecticity + adaptation* (6.12) *with accuracy $O(\epsilon)$.*

Remark 6.5. One can solve system (6.12) with any accuracy $O(\epsilon^m)$ by writing the next terms in expansion (6.18). However, the formulas obtained are very cumbersome.

Remark 6.6. In fact we see that system (6.12) is reduced to scalar equations (6.17). More exactly, the equation of zero curvature (6.12a) is reduced to the conditions $da_\gamma = 0$, $da'_\gamma = 0$. The adaptation equation (6.12a) and the symplecticity equation (6.12b) are reduced to the first and the second columns of conditions (6.17) respectively.

(vi) *Quasimodes in Adiabatic Approximation*

In the situation of previous subsection it is easy to construct the normal subbundle τ, the sections of the Floquet type (5.13) and the Hermitian connections $i\beta_\gamma$ (5.14), generated by the symplectic connection (6.18) with zero curvature. Namely, we have

$$\beta_g = \frac{1}{\epsilon} a_\gamma + a'_\gamma + O(\epsilon),$$

$$Y_\gamma = \sum_{\gamma=1}^{2r} y_\gamma^\nu X_\nu + O(\epsilon). \quad (6.19)$$

Precisely these forms β_γ are used in the quantization rule (4.21) over Λ.

Now let us present the explicit formula for quasienergies and quasimodes in the considered adiabatic approximation. Introduce certain notation. Denote

$$\mathbb{J} \overset{\text{def}}{=} \sum_{\nu,\mu=1}^{2r} \mathscr{I}^{-1\mu\nu} X_\nu \otimes X_\mu,$$

where $r \equiv \dim \mathscr{M} - \dim \Lambda$, $\mathscr{I}_{\mu\nu} \equiv \omega(X_\nu, X_\mu)$, $\{X_\mu\}$ is the basis of sections of the symplectic bundle E over Λ. We consider the morphism

$$\mathbb{H} \overset{\text{def}}{=} \mathbb{J} \cdot D^2 H|_E, \qquad \mathbb{H}(\alpha): E(\alpha) \to E(\alpha), \qquad \forall \alpha \in \Lambda, \tag{6.20}$$

Let $Y_\gamma^\circ(\alpha)$ be the normalized eigenvectors of the operators $\mathbb{H}(\alpha)$

$$\mathbb{H} Y_\gamma^\circ = i k_\gamma^\circ Y_\gamma^\circ, \qquad (Y_\gamma^\circ, Y_\gamma^\circ)_+ = 1, \tag{6.21}$$

where $\gamma = 1, ..., r$. Denote

$$\nu_\gamma^\circ \overset{\text{def}}{=} -\int_\Lambda (k_\gamma^\circ + i(\text{ad}(H)\, Y_\gamma^\circ, Y_\gamma^\circ)_+)\, d\sigma(\alpha). \tag{6.22}$$

Furthermore we consider a complex subbundle $\tau^\circ \subset {}^{\mathbf{C}}E$ generated by the sections $\{Y_\gamma^\circ \mid \gamma = 1, ..., r\}$. The real span of these sections defines the Euclidean subbundle $\varepsilon^\circ \subset \tau^\circ$. let us calculate the form μ^{ε° by means of ε° (see (viii) in Section 2) and denote

$$\mu_j \overset{\text{def}}{=} \oint_{\Gamma_j} \mu^{\varepsilon^\circ}, \qquad \Gamma_j \in \tilde{H}_1(\Lambda). \tag{6.23}$$

The numbers μ_j are integer, and they are even if Λ is orientable.

THEOREM 6.4. *Suppose the symbol H on $T^*\mathscr{M}$ smoothly depends on an additional parameter $\epsilon \in [0, 1]$. Let Λ be a connected compact isotropic submanifold in $T^*\mathscr{M}$, smoothly depending on ϵ. Suppose Λ is invariant with respect to the flow of the field* ad(H) *and possesses an invariant measure $d\sigma$ (smooth in ϵ). Also suppose that the field* ad$(H)|_\Lambda$ *is nonresonant, its frequencies c^j* (3.15) *are small (i.e. $c^j = O(\epsilon)$), the symplectic bundle E over Λ is trivial, and the morphism $\mathbb{H}$* (6.20) *on E is strongly stable. Then the operator $\hat{H}$ over the Riemann manifold $\mathscr{M}$ has the series of quasienergies*

$$\lambda_m^{(n)} = H|_\Lambda + \hbar \sum_{j=1}^{k'} (m_j + \mu_j/4 - \delta_j(\hbar))\, c^j$$
$$+ \hbar \sum_{\gamma=1}^{r} (n_\gamma + \tfrac{1}{2})\, \nu_\gamma^\circ + O(\epsilon^4 + \hbar^{3/2}).$$

Here $n_\gamma = 0, 1, 2, ...;$ $m_j = 0, \pm 1, \pm 2, ...;$ *the numeration* $\gamma = 1, ..., r$ *corresponds to the positive eigenspace of* $\mathbb{H}$ (6.21); *normal frequencies* ν_γ° *were defined in* (6.22), *"topological" integral numbers* μ_j *were defined in* (6.23), *and* $\delta_j(\hbar)$ *were defined in* (3.12).

The quasimodes corresponding to these quasienergies are the following

$$\psi_m^{(n)}(q) = \frac{1}{c} \sqrt{\frac{2^{n_1, ..., n_r}}{n_1! \cdots n_r!}} \int_\Lambda \mathscr{H}_{n_1, ..., n_r}(\xi_1, ..., \xi_r) \bigg|_{\substack{\xi_\gamma = (k_\alpha(x(\alpha, q)/\sqrt{\hbar}), \bar{V}_\gamma(\alpha))_- \\ \gamma = 1, ..., r}}$$

$$\times \exp\left\{ \frac{i}{\hbar} \Phi(\alpha, \alpha_0 \mid q) + i \int_{\alpha_0}^{\alpha} \varkappa_m^{(n)} \right\} \frac{d\sigma(\alpha)}{\sqrt{\Delta(\alpha)}}.$$

Here the polynomials $\mathscr{H}_{n_1, ..., n_r}$ *were defined in* (4.20) *and the closed* 1-*forms* $\varkappa_m^{(n)}$ *are defined as follows*

$$\varkappa_m^{(n)} = \sum_{j=1}^{k'} \left((m_j - \delta_j(\hbar)) + \sum_{\gamma=1}^{r} \left(n_\gamma + \frac{1}{2} \right) \beta_{\gamma j} \right) \zeta^j - \frac{1}{2} \sum_{\gamma=1}^{r} \beta_\gamma - \frac{\pi}{2} \mu^{\varepsilon^\circ},$$

where ζ^j *and* β_γ *were defined in* (3.10) *and in* (6.19), $\beta_{\gamma j} \stackrel{\text{def}}{=} (1/\pi) \oint_{\Gamma_j} \beta_\gamma$, $\Gamma_j \in \widetilde{H}_1(\Lambda)$.

APPENDIX A: ANALOGS OF THE MASLOV CLASS AND THE ARNOLD FORM IN A NON-LAGRANGIAN CASE

Below we generalize in a natural way the well-known "Lagrangian" constructions of the Maslov class [3, 37, 60, 66, 81]. The non-Lagrangian (GL-Grassmanian) case was considered by P. Dazord [18]. In this Appendix we shall follow [43, 50].

Let $(\mathfrak{X}, \omega)$ be a symplectic manifold and $\Lambda \subset \mathfrak{X}$ be a certain submanifold (non-Lagrangian and nonisotropic in general).

A submanifold Λ will be called *almost polarizable* if there is a Lagrangian subbundle in $T_\Lambda \mathfrak{X}$.

Of course, any Lagrangian submanifold is almost polarizable automatically. Also, an arbitrary submanifold in $\mathfrak{X} = T^*\mathscr{M}$ is almost polarizable.

LEMMA A.1. *If* Λ *is almost polarizable*, *then there is the Kählerian subbundle in* ${}^{\mathbf{C}}T_\Lambda \mathfrak{X}$.

Now let $L', L'' \subset T_\Lambda \mathfrak{X}$ be two subbundles of *half dimension*

$$\dim L' = \dim L'' = \tfrac{1}{2} \dim \mathfrak{X}$$

Let Π be an anti-Kählerian subbundle. As in [43], we consider

$$J = \frac{\det \omega(u' \overset{\otimes}{,} X)}{\det \omega(u'' \overset{\otimes}{,} X)}$$

the *Jacobian of projection* ${}^{\mathbf{C}}L'$ *onto* ${}^{\mathbf{C}}L''$ *along* Π with respect to arbitrary local bases of sections

$$u' = \{u'_j\} \quad \text{in } L', \qquad u'' = \{u''_j\} \quad \text{in } L'', \qquad X = \{X^j\} \quad \text{in } \Pi.$$

The map

$$\frac{J^2}{|J|^2} : \Lambda \to \mathbf{S}^2 \subset \mathbf{C}$$

does not depend on the choice of the bases and is defined as a global smooth map.

Consider the fundamental 1-form $\phi = dz/2\pi i z$ in $\mathbf{S}^1 = \{|z| = 1\}$ and denote by

$$\mu(L', L'') = -\left(\frac{J^2}{|J|^2}\right)^* \phi \tag{A.1}$$

the corresponding closed 1-form on Λ. This is an analog of the Arnold form [3].

THEOREM A.1. *Let Λ be an almost polarizable submanifold in a symplectic $\mathfrak{X}$. Fix an anti-Kählerian subbundle $\Pi \subset {}^{\mathbf{C}}T_\Lambda \mathfrak{X}$. Then for any half-dimensional subbundles $L', L'' \subset T_\Lambda \mathfrak{X}$ a closed* 1*-form* (*A*.1) *is defined such that*

(i) $\mu(L', L'') = -\mu(L'', L')$;

(ii) $\mu(L', L'') + \mu(L'', L''') + \mu(L''', L') = 0$;

(iii) *the cohomology class*

$$m(L', L'') \overset{\text{def}}{=} [\mu(L', L'')] \in H^1(\Lambda, \mathbf{Z})$$

is integral and independent on Π; it also possesses properties (i), (ii);

(iv) $m(L', L'') = w_1(L') + w_1(L'') \pmod 2$, *where w_1 denotes the Steifel–Whitney class.*

If there is an isomorphism $g: L' \to L''$ identical on the base Λ, then $m(L', L'') = 0 \pmod 2$.

We see that the class $m(L', L''')$ is defined not only for Lagrangian Λ and for Lagrangian L', L''. Of course, if Λ is a Lagrangian submanifold in $\mathfrak{X} = T^*\mathcal{M}$, $L' = T\Lambda$ and $L'' = \mathcal{P}$, then $m(T\Lambda, \mathcal{P})$ is the ordinary Maslov class.

And now we shall give another equivalent definition.

A subbundle in ${}^{\mathbf{C}}T_\Lambda\mathfrak{X}$ will be called *real* if the transition matrices between any local bases in this subbundle are real.

Fix an anti-Kählerian subbundle $\Pi \subset {}^{\mathbf{C}}T_\Lambda\mathfrak{X}$.

Lemma A.2. *Half-dimensional subbundles $L \subset T_\Lambda\mathfrak{X}$ correspond one-to-one to real half-dimensional subbundles $\mathscr{E} \subset \Pi$. Namely*

$$\mathscr{E} = \textit{projection } L \textit{ on } \Pi \textit{ along } \bar{\Pi}, \tag{A.2}$$

or

$$L = \operatorname{Re} \mathscr{E}.$$

For any pair $\mathscr{E}', \mathscr{E}'' \subset \Pi$ define

$$\nu(\mathscr{E}', \mathscr{E}'') = \frac{1}{\pi} \operatorname{Im} \operatorname{tr}[dM \cdot M^{-1}], \tag{A.3}$$

where

$$M \stackrel{\text{def}}{=} \frac{(v' \overset{\otimes}{,} v'')_-}{(v'' \overset{\otimes}{,} v'')_-},$$

$v' = \{v'_i\}$ is a local basis in $\mathscr{E}'$; $v'' = \{v''_i\}$ is a basis in $\mathscr{E}''$, and the inner product $(..., ...)_-$ was defined in (2.1).

Lemma A.3. *Formula* (A.3) *defines correctly a global closed* 1-*form on* Λ. *We have*

(i) $\nu(\mathscr{E}', \mathscr{E}'') = \mu(L', L'')$ *if* $L' = \operatorname{Re} \mathscr{E}'$, $L'' = \operatorname{Re} \mathscr{E}''$,

(ii) *for any automorphism* $g: \Pi \to \Pi$

$$\nu(g\mathscr{E}', \mathscr{E}'') = \frac{1}{2\pi} \operatorname{Im} d(\ln(\det g)^2) + \nu(\mathscr{E}', \mathscr{E}'').$$

In particular, if $g^2 \in \mathbf{SL}(\Pi)$, *then*

$$\nu(g\mathscr{E}', \mathscr{E}'') = \nu(\mathscr{E}', \mathscr{E}'').$$

This Lemma is almost obvious. And Theorem A.1 is a simple corollary of it. Now we specify these results for Lagrangian and isotropic subbundles.

LEMMA A.4. (a) *Lagrangian subbundles* $L \subset T_\Lambda \mathfrak{X}$ *and Euclidean subbundles* $\mathscr{E} \subset \Pi$ *are in the one-to-one correspondence*; *see* (A.2).

(b) *Let* $\Lambda \subset \mathfrak{X}$ *be an isotropic submanifold,* $\mathscr{P} \subset T_\Lambda \mathfrak{X}$ *be a certain Lagrangian subbundle, and* $\tau \subset {}^{\mathbf{C}}T_\Lambda \mathfrak{X}$ *be the normal positive subbundle over* Λ (*see subsection* (ii) *Section* 2; *here we identify* $\bar{\rho} \approx \tau$). *Suppose there exists an Euclidean subbundle* $\varepsilon \subset \tau$. *Then* $L^\varepsilon =^{\text{def}} T\Lambda \oplus \mathrm{Re}(\varepsilon)$ *is a real Lagrangian subbundle and the form* $\mu^\varepsilon =^{\text{def}} \mu(L^\varepsilon, \mathscr{P})$ *defines the integral class*

$$[\mu^\varepsilon] \in H^1(\Lambda, \mathbf{Z}).$$

We have

$$[\mu^\varepsilon] = w_1(\Lambda) + w_1(\varepsilon) + w_1(\mathscr{P}) \pmod 2.$$

(c) *If, in addition to* (b), *there is a symplectic subbundle* $E \subset T_\Lambda \mathfrak{X}$ *such that* $\tau \subset {}^{\mathbf{C}}E$, *and there exists a decomposition* $\mathscr{P} = \mathscr{P}_1 \oplus \mathscr{P}_2$, *where* $\mathscr{P}_1 \subset E^Y$, $\mathscr{P}_2 \subset E$, *then*

$$\mu^\varepsilon = \mu(T\Lambda, \mathscr{P}_1) + \mu(\mathrm{Re}(\varepsilon), \mathscr{P}_2).$$

In this case the integral class $[\mu^\varepsilon]$ *is the sum of two classes*

- *the Maslov class of* $T\Lambda$ *in* E^Y

and

- *the Maslov class of* $\mathrm{Re}(\varepsilon)$ *in* E.

Remark A.1. If Lagrangian subbundles L^ε and $\mathscr{P}$ are transversal, i.e. $L^\varepsilon \cap \mathscr{P} = \{0\}$, then $\mu^\varepsilon = 0$, and hence $[\mu^\varepsilon] = 0$.

APPENDIX B: FIRST VARIATION EQUATION OVER ISOTROPIC SUBMANIFOLD

Suppose $\mathfrak{X}$ is a symplectic manifold with closed nondegenerate 2-form ω. Let $\mathrm{ad}(H)$ denote the Hamiltonian field on $\mathfrak{X}$ corresponding to the function H, and

$$\omega(\mathrm{ad}(H_1), \mathrm{ad}(H_2)) = \{H_1, H_2\}_{\mathfrak{X}}$$

is the Poisson bracket of two functions H_1, H_2 on $\mathfrak{X}$.

Introduce a symplectic structure $\delta\omega$ on the tangent bundle $T\mathfrak{X}$ in the following way. Denote by ξ^j the local coordinate of the point $\xi \in \mathfrak{X}$ and by

v^j the coordinates of the vector $v \in T_\xi \mathfrak{X}$ with respect to the basis $\{\partial/\partial\xi^j\}$. Suppose the form ω is defined at the point ξ by the tensor $\omega(\xi)$, i.e.,

$$\omega = \tfrac{1}{2} \sum_{k,\, j=1}^{2d} \omega_{kj}(\xi)\, d\xi^j \wedge d\xi^k, \qquad 2d \equiv \dim \mathfrak{X}.$$

Then the form $\delta\omega$ at the point $(\xi, v) \in T\mathfrak{X}$ will be defined as follows [27, 65]

$$\delta\omega = \sum_{k,\, j=1}^{2d} (\omega_{kj}(\xi)\, dv^j \wedge d\xi^k + \tfrac{1}{2} v(\omega_{kj}(\xi))\, d\xi^j \wedge d\xi^k). \tag{B.1}$$

This formula gives a global closed nondegenerate 2-form on $T\mathfrak{X}$.

If $\mathscr{H}_1$ and $\mathscr{H}_2$ are two functions on $T\mathfrak{X}$, then their Poisson bracket has the form

$$\{\mathscr{H}_1, \mathscr{H}_2\}_{T\mathfrak{X}} = \langle \omega^{-1*}\, \partial_v \mathscr{H}_1, \partial_\xi \mathscr{H}_2 \rangle + \langle \omega^{-1*}\, \partial_\xi \mathscr{H}_1, \partial_v \mathscr{H}_2 \rangle$$
$$+ \langle \omega^{-1} v(\omega)\, \omega^{-1}\, \partial_v \mathscr{H}_1, \partial_v \mathscr{H}_2 \rangle.$$

We note that to each function H on $\mathfrak{X}$ a function $H^{\#}$ on $T\mathfrak{X}$ is assigned by the formula

$$H^{\#}(\xi, v) \stackrel{\text{def}}{=} \langle dH(\xi), v \rangle, \qquad v \in T_\xi \mathfrak{X}.$$

The Hamiltonian field of this function on $T\mathfrak{X}$ will be denoted by $\operatorname{ad}^{\#}(H)$. It is easy to calculate that

$$\operatorname{ad}^{\#}(H)_{(\xi,\, v)} = \operatorname{ad}(H)_\xi + \operatorname{Var}_\xi(H)\, v \cdot \frac{\partial}{\partial v}, \tag{B.2}$$

where $\operatorname{Var}_\xi(H)$ denotes the *matrix of the first variation equation* for the field $\operatorname{ad}(H)$ at the point ξ with respect to the basis $\{\partial/\partial\xi^j\}$. This matrix is defined as follows

$$\left[\frac{\partial}{\partial\xi^j}, \operatorname{ad}(H)\right] = \sum_{l=1}^{2d} \operatorname{Var}_\xi(H)^l_j \frac{\partial}{\partial\xi^l}.$$

LEMMA B.1. (a) *The trajectories $(\xi(t), v(t))$ of the field $\operatorname{ad}^{\#}(H)$ on $T\mathfrak{X}$ are represented by the trajectories $\xi(t)$ of the field $\operatorname{ad}(H)$ on $\mathfrak{X}$ and by the trajectories $v(t)$ of the variation equation*

$$\frac{d}{dt} v(t) = \operatorname{Var}_{\xi(t)}(H)\, v(t).$$

If $v(t)$ *is the solution of this equation, then* $v(t) = d\gamma_H^t v(0)$, *where* γ_H^t *is the flow of the Hamiltonian field* $\mathrm{ad}(H)$ *on* $\mathfrak{X}$. *In particular, a subbundle* $\mathfrak{m} \subset {}^{\mathbf{C}}T\mathfrak{X}$ *will be invariant with respect to action of the differential of the flow* $d\gamma_H^t$ *iff* $\mathfrak{m}$ *is invariant with respect to shifts along the trajectories of the field* $\mathrm{ad}^{\#}(H)$.

(b) *If a submanifold* $\Lambda \subset \mathfrak{X}$ *is invariant with respect to the flow* γ_H^t, *then statement* (a) *holds for subbundles* $\mathfrak{m} \subset T_\Lambda \mathfrak{X}$. *In this case the matrix* ${}^{\mathfrak{m}}\mathrm{Var}(H)$ *of the first variation equation on an invariant subbundle* $\mathfrak{m}$ *is defined with respect to any* (*local*) *basis of sections* $\{b_j\} \subset \Gamma^\infty(\mathfrak{m})$ *by the formula*

$$[b_j, \mathrm{ad}(H)] = \sum_{l=1}^{m} {}^{\mathfrak{m}}\mathrm{Var}(H)_j^l \, b_l, \qquad (m = \dim \mathfrak{m}). \tag{B.3}$$

The variation equation on $\mathfrak{m}$ *can be written with respect to the coordinates* w^k *in this basis*

$$w \equiv \sum_{l=1}^{m} w^l b_l, \qquad \frac{d}{dt} w^l = \sum_{j=1}^{m} {}^{\mathfrak{m}}\mathrm{Var}(H)_j^l \, w^j.$$

About geometry of a variation equation on general symplectic manifolds see in the fundamental investigation by Marsden, Ratiu, Raugel [65].

We point out that, in the situation of Lemma B.1(b), the Hamiltonian field $\mathrm{ad}(H)^{\#}$ can be, of course, restricted to an invariant subbundle $\mathfrak{m}$. This restriction will be denoted by ${}^{\mathfrak{m}}\mathrm{ad}(H)^{\#}$. We have an analog of formula (B.2) for the restriction to $\mathfrak{m}$, and the vector $v = (v^1, ..., v^m)$ in (B.2) must be replaced by the vector $w = (w^1, ..., w^m)$ of the coordinates of a point in $\mathfrak{m}(\alpha)$ with respect to the basis $\{b_j\}$.

Further, suppose a subbundle $\mathfrak{n}$ is not invariant, but transversal to an invariant subbundle $\mathfrak{m}$, i.e.

$${}^{\mathbf{C}}T_\Lambda \mathfrak{X} = \mathfrak{n} \oplus \mathfrak{m}.$$

Then the *projection* ${}^{\mathfrak{n}}\mathrm{Var}(H)$ *of the variation equation of the field* $\mathrm{ad}(H)$ *on* $\mathfrak{n}$ *along* $\mathfrak{m}$ can be naturally defined. The matrix of this projection with respect to a certain local basis of sections $\{c_s\} \subset \Gamma^\infty(\mathfrak{n})$ can be calculated as follows

$$[c_s, \mathrm{ad}(H)] = \sum_{l} {}^{\mathfrak{n}}\mathrm{Var}(H)_s^l \, c_l + (\text{section of } \mathfrak{m}). \tag{B.4}$$

The Hamiltonian field $\mathrm{ad}^{\#}(H)$ can be also projected on $\mathfrak{n}$ along $\mathfrak{m}$. The projection will be denoted by ${}^{\mathfrak{n}}\mathrm{ad}^{\#}(H)$. An analog of formula (B.2) can be written for this projection.

In particular, if Λ is an isotropic (and invariant with respect to γ_H^t) submanifold in $\mathfrak{X}$, then the first variation equation can be defined on invariant subbundles $T\Lambda$ and $(T\Lambda)^Y$, and on their complexifications. Further, let $E \subset (T\Lambda)^Y$ be a certain symplectic subbundle. Since E is transversal to $T\Lambda$, the projection of the variation equation on E along $T\Lambda$ can be defined.

Similarly, the variation equation of the field $\mathrm{ad}(H)$ is defined on the normal symplectic bundle $\mathring{E}$ and on its complexification ${}^{\mathbf{C}}\mathring{E}$ (see Section 2). Hence one can speak about subbundles $\tau \subset {}^{\mathbf{C}}\mathring{E}$ invariant with respect to the variation equation. If $\{\mathring{X}_\gamma\}$ is a local basis of sections of such a subbundle, then the matrix of the variation equation on τ with respect to this basis is determined as follows

$$[X_\gamma, \mathrm{ad}(H)] = \sum_{\nu=1}^{r} {}^{\tau}\mathrm{Var}(H)_\gamma^\nu X_\nu + (\text{section of } {}^{\mathbf{C}}T\Lambda). \tag{B.5}$$

Here X_γ is the section of ${}^{\mathbf{C}}E$ which can be projected into $\mathring{X}_\gamma$ under the mapping (2.0), $\gamma = 1, ..., r$, $r = {}^{\mathbf{C}}\mathrm{dim}\, \tau$.

In particular, if τ is a normal positive subbundle over Λ, and if ε is a certain quasi-orientation in τ, then we can consider in (B.5) only local bases from ε. This yields the matrix of the variation equation ${}^{\varepsilon}\mathrm{Var}(H)$ *with respect to the quasi-orientation* ε.

One can also write on $\bar{\tau}$ a formula similar to (B.2). Namely, denote by $D_H + {}^{\bar{\tau}}\mathrm{ad}^{\#}(H)$ the restriction to $\bar{\tau}$ of the projection of the field $\mathrm{ad}^{\#}(H)$ obtained under the mapping (2.0). Then

$$D_H = \mathrm{ad}(H)_\alpha + {}^{\bar{\tau}}\mathrm{Var}(H)\, z \cdot \frac{\partial}{\partial z}, \tag{B.2a}$$

where z^γ are the coordinates in the fibers of $\bar{\tau}$ with respect to the basis $\{\bar{\mathring{X}}_\gamma\}$ from (B.5).

Now assume that a linear connection ∇ is given on a certain subbundle $\mathfrak{m}$ over Λ, invariant with respect to the variation equation of the field $\mathrm{ad}(H)$. Following [51] we shall call the variation equation on $\mathfrak{m}$ *adaptable* to the connection ∇ (or the connection will be called *adapted* to the variation equation), if the translation along the trajectories of this equation coincides with the parallel transport by the connection ∇ along the trajectories of the field $\mathrm{ad}(H)$.

Analytically, this means that

$$^{\mathfrak{m}}\mathrm{Var}(H) = -\langle \theta, \mathrm{ad}(H) \rangle, \tag{B.6}$$

where θ is the matrix of the connection form in that local basis with respect to which the matrix ${}^{\mathfrak{m}}\mathrm{Var}(H)$ is calculated. Otherwise, the condition (B.6) can be written as follows

$$^{\mathfrak{m}}\mathrm{ad}^{\#}(H) = {}^{\mathfrak{m}}\tilde{\nabla}_{\mathrm{ad}(H)|_{\Lambda}}. \tag{B.7}$$

Here we use the notation

$$^{\mathfrak{m}}\tilde{\nabla}_u \stackrel{\mathrm{def}}{=} u - \sum_{j,l} \langle \theta^l_j, u\rangle\, w^j \cdot \frac{\partial}{\partial w^l}, \tag{B.8}$$

where u is any vector field on Λ, and w^j are the coordinates in the fibers $\mathfrak{m}$ with respect to that local basis $\{b_j\}$, by which the matrix of the connection form θ is calculated: $\nabla b_j = \sum_l \theta^l_j b_l$. Note that the operator (B.8) is independent of the choice of the basis and defines a *covariant derivative in the bundle* $\mathrm{Pol}(\mathfrak{m})$ *over* Λ, *whose fibers are the spaces of polynomials on the fibers of* $\mathfrak{m}$.

Of course, a connection can be always restricted to a subbundle invariant with respect to parallel transports. Moreover, a connection is always projected along such a subbundle. Thus the *notion of "adaptation" can be also introduced on subbundles* $\mathfrak{n}$ *transversal to invariant subbundles* $\mathfrak{m}$. The adaptation condition has the form similar to (B.6) or (B.7).

In particular, suppose a normal positive subbundle τ over Λ with Hermitian connection ∇ is invariant with respect to the projection of the variation equation on ${}^{\mathbf{C}}\mathring{E}$. Then the adaptation condition on τ has the form

$$i\,\mathrm{tr}({}^{\varepsilon}\mathrm{Var}(H)) = \langle \beta^{\varepsilon}, \mathrm{ad}(H)|_{\Lambda}\rangle. \tag{B.9}$$

Here ε is a quasi-orientation in τ, and $i\beta^{\varepsilon}$ is the trace of the matrix of connection form with respect to ε. Condition (B.9) can be also written in the form (B.7), namely,

$$^{\tau}\mathrm{ad}^{\#}(H) = {}^{\tau}\tilde{\nabla}_{\mathrm{ad}(H)|_{\Lambda}}.$$

The Hermitian connection ∇ on τ induces, in a standard way, a Hermitian connection $\bar{\nabla}$ on $\bar{\tau}$ which, in its turn, generates the operator $D_H \equiv {}^{\bar{\tau}}\tilde{\nabla}_{\mathrm{ad}(H)|_{\Lambda}}$. So we have

$$D_H = \mathrm{ad}(H)_{\alpha} - \sum_{j,l} \langle \theta^l_j, \mathrm{ad}(H)\rangle\, z^j \cdot \frac{\partial}{\partial z^l},$$

where $\{z^j\}$ are local coordinates in the fiber $\bar{\tau}$ corresponding to an arbitrary basis and θ is the corresponding matrix-valued form of connection.

The pre-image of the subbundle τ under the projection (2.0) will be denoted by $\bar{\mathscr{R}}$. Or otherwise: $\mathscr{R} = \rho \oplus {}^{\mathbf{C}}T\Lambda$, where ρ is defined in subsection (iii) Section 2.

The subbundle $\mathscr{R} \subset {}^{\mathbf{C}}T_\Lambda \mathfrak{X}$ is Lagrangian; the form $(..., ...)_-$ on $\mathscr{R}$ is nonnegative, but degenerate, its rank is equal to $\frac{1}{2}\dim \mathfrak{X} - \dim \Lambda$. A subbundle with such properties is called a *complex germ* over Λ in the theory [67].

LEMMA B.2. *Suppose τ is invariant with respect to the projection of the variation equation on the field* $\mathrm{ad}(H)$. *Then*

(a) *The subbundle $\bar{\mathscr{R}}$ is also invariant. Thus the projection ${}^{\Pi}\mathrm{Var}(H)$ of the variation equation on Π along $\bar{\mathscr{R}}$ is defined. The subbundle $\rho \subset \Pi$ is invariant with respect to this projection, i.e. the variation equation ${}^{\rho}\mathrm{Var}(H)$ is defined*;

(b) *The projection of the field* $\mathrm{ad}^{\#}(H)$ *on $\rho \oplus \bar{\rho}$ along ${}^{\mathbf{C}}T\Lambda$ is defined; its restriction to ρ is denoted by ${}^{\rho}\mathrm{ad}^{\#}(H)$. There is an analog of formula* (B.2)

$$ {}^{\rho}\mathrm{ad}^{\#}(H) = \mathrm{ad}(H)_\alpha + {}^{\rho}\mathrm{Var}(H)\, z \cdot \frac{\partial}{\partial z}, \tag{B.10}$$

where z are the coordinates of a point in the fiber ρ_α.

APPENDIX C: PROOF OF THEOREMS 3.1 AND 4.1

Here we give the proof of the main statements of Sections 3 and 4 in the case of the Euclidean phase space $\mathfrak{X}$. Let $\mathfrak{X} = (\mathbf{R}^d)_q \oplus (\mathbf{R}^d)_p$ possess the symplectic structure $\omega = dp \wedge dq$.

Suppose an isotropic submanifold $\Lambda \subset \mathfrak{X}$ is defined by equations

$$\Lambda = \left\{ \begin{pmatrix} q \\ p \end{pmatrix} \,\middle|\, q = q(\alpha),\ p = p(\alpha) \right\}.$$

The anti-Kählerian almost polarization Π(2.5) can be described as

$$\Pi = \left\{ \sum_{j=1}^{d} y^j Z_j \mid y^j \in \mathbf{C} \right\},$$

where $Z_j = Fe_j$ is a basis in Π obtained by the projection (2.8) from an Euclidean basis e_j in $(\mathbf{R}^d)_q$. We note that $\omega(e^i, Z_j) = \delta^i_j$, where e^i is the dual Euclidean basis in $\mathscr{P} = (\mathbf{R}^d)_p$. So

$$Z_j = \begin{pmatrix} e_j \\ \sum_i A_{ji} e^i \end{pmatrix}, \qquad A_{ij} = \omega(Z_i, e_j), \qquad \operatorname{Im}((A_{ij})) > 0,$$

and the almost polarization consists of planes

$$\Pi(\alpha) = \left\{ \begin{pmatrix} y \\ A(\alpha)\, y \end{pmatrix} \,\middle|\, y \in \mathbf{C}^d \approx {}^{\mathbf{C}}Q(\alpha) \right\}.$$

Thus we identify ${}^{\mathbf{C}}Q(\alpha) \overset{F_\alpha}{\approx} \Pi(\alpha)$, and consider $A(\alpha)$ as an isomorphism ${}^{\mathbf{C}}Q(\alpha) \to {}^{\mathbf{C}}\mathscr{P}(\alpha)$.

The phase (2.11) of the Gaussian packet (3.2) in this notation has the form

$$\begin{aligned} \Phi(\alpha, \alpha_0 \mid q) = \int_{\alpha_0}^{\alpha} p(\alpha) \cdot dq(\alpha) + q(\alpha) \cdot (q - q(\alpha)) \\ + \tfrac{1}{2} A(\alpha)(q - q(\alpha)) \cdot (q - q(\alpha)). \end{aligned}$$

We identify the functions $\tilde{\varphi}(\alpha, y)$ on $\Lambda \times \mathbf{C}^d$ with the functions on Π. By $\mathscr{F}(\Pi)$, we denote the space of all such functions polynomial in y. Let $\mathscr{F}_0(\Pi)$ be a subspace in $\mathscr{F}(\Pi)$ consisting of functions, whose supports on Λ lie in a simply connected domain.

Introduce on $\mathscr{F}(\Pi)$ a vector-operator $\mathbf{a} = (\mathbf{a}^1, ..., \mathbf{a}^{2d})$ by the formula

$$\mathbf{a} \overset{\text{def}}{=} \begin{pmatrix} y \\ Ay - i\partial/\partial y \end{pmatrix} \equiv \exp\left(-\frac{i}{2} Ay \cdot y\right) \circ \begin{pmatrix} y \\ -i\partial/\partial y \end{pmatrix} \circ \exp\left(\frac{i}{2} Ay \cdot y\right).$$

Then to each vector field v on $\mathfrak{X} = \mathbf{R}^{2d}$ we assign an operator

$$\mathbf{v} \overset{\text{def}}{=} \omega(v|_\Lambda, \mathbf{a}) \qquad \text{(C.1)}$$

on the space $\mathscr{F}(\Pi)$. Obviously, we have

$$[\mathbf{v}', \mathbf{v}''] = -i\omega(v', v'') \cdot I,$$

where I is the unit operator.

LEMMA C.1. *Suppose a vector field u is tangent to Λ, i.e., $u|_\Lambda \in T\Lambda$. Then for any $\tilde{\varphi} \in \mathscr{F}(\Pi)$ the following identity holds:*

$$- i\hbar u_\alpha \left(\exp\left\{\frac{i}{\hbar}\Phi(\alpha, \alpha_0 \mid q)\right\} \tilde{\varphi}(\alpha, y)\Big|_{y = q - q(\alpha)/\sqrt{\hbar}} \right)$$

$$= \exp\left\{\frac{i}{\hbar}\Phi(\alpha, \alpha_0 \mid q)\right\}$$

$$\times \left[\left(\sqrt{\hbar}\,\mathbf{u} - i\hbar u_\alpha + \frac{\hbar}{2} u_\alpha(A(\alpha))\, y \cdot y \right) \tilde{\varphi}(\alpha, y) \right]\Big|_{y = q - q(\alpha)/\sqrt{\hbar}}.$$

Here the vector field $u = u_\alpha$ acts with respect to the variable $\alpha \in \Lambda$, and the operator $\mathbf{u}$ *is assigned to the field u by the formula* (C.1).

The statement of this lemma can be verified by the direct differentiation of the function Φ in α and in q, namely:

$$u_\alpha(\Phi) = \sqrt{\hbar}\, \omega\left(u, \begin{pmatrix} y \\ Ay \end{pmatrix} \right) + \frac{\hbar}{2} u(A)\, y \cdot y,$$

$$\frac{\partial \Phi}{\partial q} = p(\alpha) + \sqrt{\hbar}\, A(\alpha)\, y,$$

where $y \equiv (q - q(\alpha))/\sqrt{\hbar}$.

Further, by using standard formulas from the perturbation theory for the functions in operators, we get the following simple fact.

Lemma C.2. *For any symbol $H(q, p)$ (i.e., for any smooth function increasing together with all its derivatives not greater than a polynomial), the operator $\hat{H} = H(q, -i\hbar\, \partial/\partial q)$ acts on the expression under the integral in* (4.5) *as follows*

$$\hat{H}\left(\exp\left\{\frac{i}{\hbar}\Phi(\alpha, \alpha_0 \mid q)\right\} \tilde{\varphi}(\alpha, y)\Big|_{y = q - q(\alpha)/\sqrt{\hbar}} \right)$$

$$= \exp\left\{\frac{i}{\hbar}\Phi(\alpha, \alpha_0 \mid q)\right\} \times \Bigg(\Bigg(H|_\Lambda - \sqrt{\hbar}\,\mathbf{ad}(\mathbf{H})$$

$$+ \frac{\hbar}{2} D^2 H|_\Lambda\, \mathbf{a} \cdot \mathbf{a} + O(\hbar^{3/2}) \Bigg) \tilde{\varphi}(\alpha, y) \Bigg)\Big|_{y = q - q(\alpha)/\sqrt{\hbar}}. \qquad \text{(C.2)}$$

Here the operator $\mathbf{ad}(\mathbf{H})$ *corresponds to the field* $\mathrm{ad}(H)$ *by* (C.1), *and D^2H denotes the matrix of second derivatives of the function H in Euclidean coordinates on $\mathfrak{X} = (\mathbf{R}^d)_q \oplus (\mathbf{R}^d)_p$, i.e., $D^2 H = \begin{pmatrix} H_{qq} & H_{qp} \\ H_{pq} & H_{pp} \end{pmatrix}$.*

Now the summand $-\sqrt{\hbar}\exp\{i\Phi/\hbar\}\, \mathbf{ad}(\mathbf{H})\tilde{\varphi}$ in the right-hand side of (C.2) will be transformed by integrating by parts on Λ with respect to the measure $d\sigma$. For this purpose let us consider an integral operator of type (4.5)

$$\mathbb{I}_{\mathbf{1}}\colon \mathscr{F}(\Pi) \to C_h^\infty(\mathbf{R}^d),$$
$$\mathbb{I}_{\mathbf{1}}(\tilde{\varphi})(q) \stackrel{\text{def}}{=} \frac{1}{c}\int_\Lambda \exp\left\{\frac{i}{h}\Phi(\alpha, \alpha_0 \mid q)\right\} \tilde{\varphi}\left(\alpha, \frac{q-q(\alpha)}{\sqrt{h}}\right) d\sigma(\alpha), \tag{C.3}$$

where the constant c is defined in (3.4)

$$c = 4^{k/4}(\pi h)^{(k+d)/4}, \qquad k \equiv \dim \Lambda.$$

LEMMA C.3. *If the field* ad(H) *is tangent to* Λ *and the measure* $d\sigma$ *on* Λ *is invariant with respect to the flow of this field, then for any* $\tilde{\varphi} \in \mathscr{F}_0(\Pi)$, *the commutation formula holds*

$$\hat{H}\mathbb{I}_{\mathbf{1}}(\tilde{\varphi}) = \left(H|_\Lambda \cdot \tilde{\varphi} - ih \operatorname{ad}(H)\, \tilde{\varphi} + \frac{h}{2}(\operatorname{ad}(H)(A)\, y\cdot y + D^2 H \mathbf{a}\cdot\mathbf{a})\, \tilde{\varphi}\right)$$
$$+ O(h^{3/2}). \tag{C.4}$$

Remark C.1. Here and below the field ad(H) and the matrix D^2H are assumed to be restricted on Λ, but we do not mark this explicitly.

Now let us transform the right-hand side of (C.4). The third summand in this formula has the form

$$\operatorname{ad}(H)(A)\, y\cdot y + D^2H\mathbf{a}\cdot\mathbf{a} = 2iKy\cdot\frac{\partial}{\partial y} - H_{pp}\frac{\partial}{\partial y}\cdot\frac{\partial}{\partial y} + My\cdot y + i\operatorname{tr}(K).$$

Here the following notation are introduced

$$K \stackrel{\text{def}}{=} -(H_{pq} + H_{pp}A),$$
$$M \stackrel{\text{def}}{=} \operatorname{ad}(H)(A) + AH_{pq} + H_{qp}A + AH_{pp}A + H_{qq}.$$

The matrices $(-K)$ and $(-M)$ are the blocks of the whole variation matrix of the field ad(H); namely,

$$[\operatorname{ad}(H), Z_i] = \sum_{l=1}^{d} (K_i^l Z_l + M_{li} e^l), \tag{C.5}$$

In the right-hand side, we have used the symplectic basis $\{Z_i(\alpha), e^j\}$ in ${}^{\mathbf{C}}T_\alpha\mathfrak{X} \approx \mathbf{C}^{2d}$ corresponding to the decomposition

$${}^{\mathbf{C}}T_\alpha\mathfrak{X} = \Pi(\alpha) \oplus {}^{\mathbf{C}}\mathscr{P}(\alpha).$$

One can also use another decomposition

$$^{\mathbf{C}}T_\alpha \mathfrak{X} = \Pi(\alpha) \oplus \bar{\mathscr{R}}(\alpha),$$

where $\mathscr{R} \overset{\text{def}}{=} \rho \oplus {}^{\mathbf{C}}T\Lambda$ (see Appendix B). A symplectic basis corresponding to this decomposition has the form $\{Z_i, Y^j\}$, where the vectors $Y^j(\alpha)$ are obtained from the orts e^j by the projection

$$^{\mathbf{C}}\mathscr{P}(\alpha) \to \bar{\mathscr{R}}(\alpha) \qquad \text{along} \quad \Pi(\alpha).$$

So we have $\omega(Y^i(\alpha), Z_j(\alpha)) = \delta^i_j$, and

$$Y^j(\alpha) = \sum_{l=1}^{d} T^{jl}(\alpha)\, Z_l(\alpha) + e^j, \tag{C.6}$$

where $T(\alpha)$ is a symmetric matrix of projection

$$T\colon \bar{\mathscr{R}} \to \Pi \qquad \text{along} \quad {}^{\mathbf{C}}\mathscr{P}.$$

with respect to bases $\{Y^j\}$ and $\{Z_i\}$.

Denote by $\mathfrak{i}\colon \Pi^* \to \bar{\mathscr{R}}$ an isomorphism given by (see Section 4):

$$(u, \overline{\mathfrak{i}(\alpha)})_- = \xi \cdot u \qquad \forall \xi \in \Pi^*_\alpha, \qquad u \in \bar{\mathscr{R}}_\alpha,$$

and then define a symmetric bivector field $\mathfrak{t}\colon \Pi^* \to \Pi$ as

$$\mathfrak{t} = T \cdot \mathfrak{i}. \tag{C.7}$$

We recall (see Lemma B.2) that if the subbundle τ is invariant with respect to the variation equation of the field $\operatorname{ad}(H)$, then the subbundle $\bar{\mathscr{R}}$ is also invariant. We denote by Γ the matrix of the variation equation on $\bar{\mathscr{R}}$ with respect to the basis $\{Y^j\}$. Then the matrix of projection of the variation equation on Π along $\bar{\mathscr{R}}$ will be: $-\Gamma^*$ in the basis $\{Z_j\}$. On the other hand, by (C.5), (C.6), this projection matrix is equal to $T \cdot M - K$. Thus we have the identity

$$K - T \cdot M = \Gamma^*$$

(the asterisk denotes the transposition without complex conjugation). This yields

Lemma C.4. *In addition to the conditions of Lemma C.3, suppose that the subbundle τ is invariant with respect to the variation equation of the field $\operatorname{ad}(H)$ over Λ. Then on $\mathscr{F}(\Pi)$, the commutation formula holds*

$$\hat{H} \cdot \mathbb{I}_{\mathbf{1}} = \mathbb{I}_{\mathbf{1}} \cdot (a_H - i\hbar b_H) + O(\hbar^{3/2}),$$

where

$$a_H \stackrel{\text{def}}{=} H|_\Lambda,$$

$$b_H \stackrel{\text{def}}{=} {}^\Pi\mathrm{ad}^\#(H) - \frac{i}{2}(H_{pp} - TMT)\frac{\partial}{\partial y}\cdot\frac{\partial}{\partial y} - \frac{1}{2}\,\mathrm{tr}(\Gamma).$$

We recall that ${}^\Pi\mathrm{ad}^\#(H)$ denotes the projection of the Hamiltonian field $\mathrm{ad}^\#(H)$ on Π along $\bar{\mathscr{R}}$ (see Appendix B). Thus we have ${}^\Pi\mathrm{ad}^\#(H) = \mathrm{ad}(H)_\alpha - \Gamma^* y\cdot\partial/\partial y$.

In the second-order operator b_H, we can annihilate the second derivatives by using the relation

$$\mathrm{ad}(H)\,T + \Gamma^* T + T\Gamma + TMT = H_{pp}.$$

Lemma C.5. *Let the symbol H be compatible with* $(\Lambda, \sigma, \tau, \nabla)$ *(see subsection* (iii) *Section* 2). *Then we have*

$$b_H = \Delta^{1/2}\cdot\exp(-\breve{\mathfrak{t}})\cdot\left({}^\Pi\mathrm{ad}^\#(H) + \frac{i}{2}\langle\varkappa, \mathrm{ad}(H)\rangle\right)\cdot\exp(\breve{\mathfrak{t}})\cdot\Delta^{-1/2},$$

where

$$\breve{\mathfrak{t}} = \frac{1}{4}\,\mathfrak{t}\,\partial\cdot\partial = -\frac{i}{2}\,T(\alpha)\frac{\partial}{\partial y}\cdot\frac{\partial}{\partial y},$$

the isomorphism $\mathfrak{t}$ *is given by* (C.7), $y = \{y^i\}$ *are the coordinates in fibers* $\Pi(\alpha)$ *with respect to the basis* $\{Z_i\}$.

We define a new integral operator over Λ by the formula

$$\mathbb{I}_2 \stackrel{\text{def}}{=} \mathbb{I}_1\cdot\Delta^{1/2}\cdot\exp(-\breve{\mathfrak{t}}). \tag{C.8}$$

Lemmas C.4 and C.5 imply that *under the condition that* $(\Lambda, \sigma, \tau, \nabla)$ *are compatible with H, the commutation formula holds*

$$\hat{H}\cdot\mathbb{I}_2 = \mathbb{I}_2\cdot\left(H|_\Lambda - ih\,{}^\Pi\mathrm{ad}^\#(H) + \frac{h}{2}\langle\varkappa, \mathrm{ad}(H)\rangle\right) + O(h^{3/2}). \tag{C.9}$$

Now in the right-hand side of (C.9) we shall restrict the field ${}^\Pi\mathrm{ad}^\#(H)$ to the subbundle ρ, i.e., we shall change this field by ${}^\rho\mathrm{ad}^\#(H)$ following (B.10). It turns out that this procedure can be carries out with accuracy $O(h^{3/2})$.

More precisely, consider the projection (4.2) and the corresponding map of function spaces

$$^{\rho}\mathfrak{f}^*\colon \mathscr{F}(\rho)\to\mathscr{F}(\Pi).$$

We denote

$$^{\rho}\mathfrak{t}={}^{\rho}\mathfrak{f}\cdot\mathfrak{t}\cdot{}^{\rho}\mathfrak{f}^*.$$

This is a symmetric bivector on ρ, so the following differential operator

$$^{\rho}\check{\mathfrak{t}}=\frac{1}{4}\,{}^{\rho}\mathfrak{t}(\alpha)\,\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial z}$$

is defined correctly on $\mathscr{F}(\rho)$ (here z denoted the coordinates in the fibers $\rho(\alpha)$ with respect to any basis of sections, the matrix $^{\rho}\mathfrak{t}(\alpha)$ is calculated with respect to the same basis; the operator $^{\rho}\check{\mathfrak{t}}$ is the independent of the choice of the basis).

Lemma C.6. *The following relations hold*

$$\check{\mathfrak{t}}\cdot{}^{\rho}\mathfrak{f}^*={}^{\rho}\mathfrak{f}^*\cdot{}^{\rho}\check{\mathfrak{t}},$$
$$^{\Pi}\mathrm{ad}^{\#}(H)\cdot{}^{\rho}\mathfrak{f}^*={}^{\rho}\mathfrak{f}^*\cdot({}^{\rho}\mathrm{ad}^{\#}(H)+g(H)),$$

where the remainder $g(H)$ satisfies the estimate

$$\mathbb{I}_2(g(H)\,\varphi)=O(\hbar^{1/2})\qquad\forall\varphi\in\mathscr{F}_0(\rho).$$

Moreover, we have

$$b=\circ\cdot{}^{\rho}\mathfrak{t}\cdot\circ^*$$

(*see Section* 4(i)).

We define a new integral operator

$$\mathbb{I}_3\overset{\mathrm{def}}{=}\mathbb{I}_2\cdot{}^{\rho}\mathfrak{f}^*\equiv\mathbb{I}_1\cdot\Delta^{1/2}\cdot{}^{\rho}\mathfrak{f}^*\cdot\exp(-{}^{\rho}\check{\mathfrak{t}}).$$

By (C.9) and Lemma (C.6), we get the commutation formulas (4.6), (4.7) on $\mathscr{F}_0(\rho)$:

$$\hat{H}\cdot\mathbb{I}_3=\mathbb{I}_3\cdot\left(H|_{\Lambda}-i\hbar\,{}^{\rho}\widetilde{\nabla}_{\mathrm{ad}(H)}+\frac{\hbar}{2}\langle\varkappa,\mathrm{ad}(H)\rangle\right)+O(\hbar^{3/2}).$$

Of course $^{\rho}\widetilde{\nabla}_{\mathrm{ad}(H)}={}^{\tau}\widetilde{\nabla}_{\mathrm{ad}(H)}\equiv D_H$. So we obtain the main commutation formula (4.6).

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